VOLUME 3

AANKALAN

THE ANNUAL JOURNAL

DEPARTMENT OF MATHEMATICS | HANSRAJ COLLEGE

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From the Principal's Desk

To achieve and promote excellence in publications and research, the college takes the initiative to unfold the third edition of its yearly academic journal of the Mathematics Department, Aankalan 2022, exclusively to publish students' research papers and articles. I feel enraptured to witness a perfect amalgamation of academic commitment and professional competency among these juveniles who have their hands behind the success of this journal. The Mathematics Department of Hansraj College has forever focused on discovering, developing and drawing out the magic lying dormant inside these young budding mathematicians. This is an add-on to the enriched catalogue of college publications and academic literature and opens a gateway to the research methodologies at the undergraduate level. The department's entire editorial staff has worked tirelessly, and I would like to adulate their perseverant efforts.
With this I would also like to give a special mention to Prof. Preeti Dharmarha who has always been a right hand for the success of this. Under her enlightenment and mentorship, the students found the courage and guidance to overcome their shortcomings.
Heartfelt wishes for the upcoming laurels with the release of Aankalan and hope that it continues to serve as the symbol of the Department's integrity and grandeur!

Dr. Rama
Principal
Hansraj College,
University of Delhi

From the Head of Faculty Advisors

Enthralled by the awesome response to the two issues of *Aankalan*, my student commanders have worked against all odds towards yet another accolade. We are so very excited to present this third edition of the Academic Publication of the Department of Mathematics, Hansraj College in 2022. And what better day for release of this edition than today, March 28, 2022, coinciding with the farewell to the outgoing batch of our department!!

Kudos to all the team members; Editor in Chief: Daksh Dheer along with Associate Editors: Khyati Khera, Gamini Magar and Dushyant Kumar Rohilla, Assistant Editors: Lakshmi P, Simran Barwa, Saloni Juneja and Prachi Sinha for their untiring efforts and showing the fighter's commitment against all odds of time, corona, physical distancing etc.

Of course, this edition would have failed to take the current shape and meet the stringent quality checks, had our reliable Advisory Editorial Board members Dr. Harjeet Arora, Mrs. Amita Aggarwal, Prof. Rakesh Batra, Dr. Mukund Madhav Mishra and Mrs. Kriti Wadhwa not given their best despite very short notice. Backing of Principal, Dr. Rama was always there. We are beholden.

I wonder, till when the Corona would keep playing hide and seek with us. Before we heave a sigh of relief, news about another variant pops up, causing panic and anxiety all over, putting a spoke in the wheel of education, economic recovery and stability. During these trying times, let us always remember, and strive to imbibe the seven secrets of success, which great Dr. APJ Abdul Kalam found in his room in our lives:

1. Roof said: Aim high
2. Fan said: Be cool
3. Clock said: Every minute is precious
4. Mirror said: Reflect before you act
5. Window said: See the world
6. Calendar said: Be up-to-date
7. Door said: Push hard to achieve your goals

The contents of this edition have also been selected after a careful review by experienced faculty. The issue comprises of a rainbow of articles encompassing 120 is Not That Simple, Beauties That Make π Unique, Understanding Mathematical Knots, Hilbert's Hotel and Infinity ∞, The Family of Metallic Ratios, On the Calculus of Variations, Decrypting the Mathematics of Cryptography, On the Elegant Utitily of Generating Functions, Calculating π Upto Higher Precision, Conway's Game of Life, Grandi's Series, Evolution of Number System and Mathematics in Map Projection. Apart from these, The Cardinalities of Infinite Sets and an IMO Problem on Anti Pascal Triangle is also discussed.

It is very encouraging for team *Aankalan* to see the overwhelming response to our call for papers for this third edition. I look forward to seeing this journal getting recognized in the maths fraternity in very near future. We have many plans in mind, starting with getting an ISSN and expanding the domain.

Till I divulge all the secrets, keep exploring, keep refining, keep learning; and enjoy this edition. Goes without saying, suggestions/ comments/ areas for improvements are very heartily welcomed.

Best wishes to all.

Prof. (Dr.) Preeti Dharmarha
Department of Mathematics,
Hansraj College

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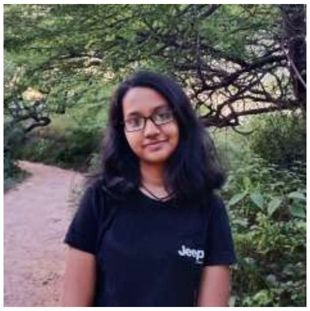

Lakshmi P

Prachi Sinha

Saloni Juneja

Simran Barwa

Team Aankalan

From the Editors' Quill

Aankalan, the Annual Mathematics Journal of Hansraj College, started off in 2019 as an attempt to spark an interest for research at an undergraduate level, to provide encouragement and a platform to write research papers and learn expository skills alongside it. Aankalan aims to apprise its readers, as well its contributors, about the richness and omnipresence of this beautiful subject. The Department has always encouraged this spirit of exploration, a spirit of curiosity and experimentation, which has manifested itself as *Aankalan.*

If one looks it up, the 'definition' of mathematical beauty comes up as "aesthetic pleasure typically derived from the abstractness, purity, simplicity, depth or orderliness of mathematics." While that is a good starting line, we believe it is in no way a complete or an all-encompassing thread that binds all mathematics, because mathematics, by virtue of its own nature, is free and unbound. Symmetry scents the air, it is a sight of beauty. Russel very aptly described mathematics as possessing not only truth, but supreme beauty - for what could be prettier than the art of patterns and symmetries that embed themselves in the fabric of the universe.

What transforms study to research is primarily the rigor of originality, and the discourse that follows from reading one another's thoughts and work. Our dream with *Aankalan* is to bridge this gap, and encourage team research and academic discourse, away from the usual individualistic nature of a typical undergraduate degree.

This year's edition, the longest and most comprehensive one yet, features papers ranging from pure mathematics including group theory, knot theory, set theory all the way to the applications of mathematics to map projection, the calculus of variations, decryption and cryptography - quite a diverse collection. The edition also includes quite a few intriguing papers, including an introduction to the classic Conway's Game of Life, an elegant problem that appeared in the IMO, with an even more elegant solution, to top it off. We hope you enjoy perusing this beautiful selection of papers.

"*O blinding hour, O holy, terrible day,*
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare."

∼ Edna St. Vincent Millay

Acknowledgements

We express our immense gratitude towards our Principal, Dr. Rama, who has always been a source of encouragement, motivation, and inspiration for all of us, as well as a staunch supporter of all our attempts to develop and progress.

A warm gratitude towards Dr. Neena Gupta, for taking time out of her busy schedule to share her experiences and impart knowledge with us. The Department will always be grateful to her.

We are also highly grateful to all the Department's teachers for their unwavering support and advice. We would like to express our gratitude to Dr. Preeti Dharmarha, Dr. Harjeet Arora, Ms. Amita Aggarwal, Dr. Mukund Madhav Mishra, Dr. Rakesh Batra, and Dr Kriti Wadwha (TIC) for their comprehensive and detailed input on the articles, without which the publication of the Journal would not have conceivable.

We would like to express our gratitude to the Department's President, Anvesha Kushwah, for constantly pointing us in the correct road. We would also like to express our gratitude to the Department's Vice Presidents, Khushi Agarwal and Sanskriti Jha, for their unwavering support and assistance. Furthermore, we thank the former Aankalan team, including former Editor-in-Chief, Utcarsh Mathur, and former Associate Editors, Adityendra Tiwari, Apurva Chauhan, Shivam Belvanshi and Samarth Rajput, for believing in us to carry the legacy forward and assisting us in every way possible.

We thank the technical team for helping us design the cover of *Aankalan* and in providing other assistance. We also thank the Developer team for their arduous efforts on the Department website that gave *Aankalan* a platform. We also thank all members of the Mathematics Department Council for their constant support and encouragement.

Last of all, but certainly not least, we applaud the Department's students for their continual efforts to propel the Department forward and for their tremendous participation and involvement this year and helping us create the most comprehensive edition of *Aankalan* yet.

We thank you all for your support!

Interview with Dr. Neena Gupta

The Aankalan team, Department of Mathematics, had the absolute privilege of hosting none other than **Dr. Neena Gupta**, *Ramanujan Prize Awardee 2021*, for an interview.

Dr. Neena Gupta is one of the most well-known personalities in the field of mathematics. She is a professor at the Indian Statistical Institute's (ISI) Statistics and Mathematics Unit in Kolkata. Her main research interests are commutative algebra and affine algebraic geometry. She was awarded the Shanti Swarup Bhatnagar Prize for Science and Technology (2019) in the category of mathematical sciences, India's highest scientific and technological honour. Her solution to the *Zariski Cancellation Problem* earned her the 2014 Indian National Science Academy Young Scientists Award, as well as made her a familiar name in the academic space within and outside the country. The Indian National Science Academy described her solution as "one of the best works in algebraic geometry in recent years done anywhere". Dr. Gupta is also the 2021 recipient of the DST-ICTP-IMU Ramanujan Prize for Young Mathematicians from Developing Countries. She is the third woman from India to receive this honour.
She received her post-graduate degree in mathematics from the Indian Statistical Institute in 2008 and her Ph.D. in 2011 with commutative algebra as her specialization. Her dissertation was titled "Some Results on Laurent Polynomial Fibrations and Quasi A-algebras."
Being able to interact with her was quite the motivation for all of us. We present here a few excerpts from the interview conducted by Assistant Editors **Lakshmi P** and **Saloni Juneja**.

Saloni: We would like to embark upon the conversation with you telling us something about your journey - from not wanting to continue pursuing mathematics in graduation, all the way to winning one of the most prestigious awards: the Ramanujan Prize.

Dr. Gupta: This information that I did not want to pursue mathematics is not correct. In fact, from childhood itself I had a huge passion for mathematics, it was my life. In high school when most of my peers were going for engineering or medicine, I met a senior doing math honours and realised it's for me. But what was not clear to me during the time of my graduation was that there was a career and a life as a researcher in mathematics which I could pursue. But the fact that I will be with mathematics, that was something I always knew.

Lakshmi: You've mentioned before how you were discouraged from pursuing Zariski's cancellation problem and now seeing how everything's turned out, how do you deal with such situations? What motivated you to continue with it?

Dr. Gupta: During my Ph.D. days, a professor visited. I was reading up on some of his papers with some examples and I had some ideas which I asked the professor about. He was kind enough and suggested not to take up the problem. He said it very kindly, because as a Ph.D. scholar, I should be more inclined towards completing my thesis rather than being occupied with a problem that was very difficult. During my TIFR days, I had some free time and that's how I came back to the problem. So, the problem has 2 parts. I was able to solve it for $m \geq 2$. The other part is still an open problem. Looking back on it, it happened when I was well-prepared to handle that problem and I could solve it.

Saloni: It is a commonly held belief that academia is a male dominated space. Would you agree? Does it, in your opinion, make academia a less inclusive space for women?

Dr. Gupta: It is not, in fact. Yes, there are very few women in academia, but there's nothing preventing them from doing it. Women are not any less capable, and they are in every field right from the farms to being pilots. She can do anything, including academia. All that matters is the interest.

Saloni: Yes ma'am, exactly. Women are no less capable but still there's a minority of women. Why do you think that is?

Dr. Gupta: There may be hurdles as we see a sharp decline in numbers after graduation and post graduation in STEM careers. Maybe there's a pressure to take up more lucrative jobs or to settle soon, and it takes quite a long time to settle down in a research career. In my case, I had a supportive family. My father, for instance, when I told him I wanted to do a Ph.D., he asked me how many years would it take. I said it takes us 5 years, but people have finished in 3 years as well. He replied, stating that I'd definitely finish in 2 years, and it was with his encouragement and blessings that I completed the research part of my Ph.D. in 2 years. So, the point here is that it takes a lot of time getting started and that's probably why there are so few women in this space.

Lakshmi: We look up to you and idolize you a lot. Do you see a mathematician/academic as your idol? If yes, whom and why?

Dr. Gupta: Okay so, one idol which is probably not just for me but idol for many mathematicians, including men, would be Emmy Noether. Emmy Noether was one of the greatest mathematicians. Throughout her life, she proved several mind-blowing and fundamental results in algebra and she gave birth to the theory of commutative algebra on which I'm currently working. She is also known as the mother of commutative algebra. With my professor Omarth Hemdutta, I've written an article on her. Emmy Noether is an idol that I really admire.

Saloni: As we've seen, you've worked on affine geometry and a student usually learns Euclidean and Cartesian geometry in schools, after that, at many Universities, including DU, a geometry course is introduced at a PG level. Do you think Geometry should be a part of the UG curriculum itself? If yes, what type of geometry may be a good component of a UG course in Mathematics?

Dr. Gupta: The geometry you're talking about is algebraic geometry, which already exists in courses but is probably not spelled out that way. I feel it cannot be introduced earlier because it requires quite a lot of background knowledge. The machinery itself takes a lot of time. So what be done at the UG or high school level is pre-19th or 19th century geometry, which can be taught very informally without making definitions unlike its 20th century counterpart, which went through huge changes and became very formal. Thus, introducing it in a full-fledged way is not possible. However, basic courses such as introduction to curves can be taught but even that would require a bit of background knowledge to understand.

Lakshmi: What are your thoughts on the scope of pure mathematics, specifically concepts that necessarily don't find applications in other fields?

Dr. Gupta: It is not that all the discoveries made in the field will have application in the near future but in the long term everything is going to be used. For example, Ramanujan used to work with continued fractions, powers and expansions, all of which is pure mathematics and today, after a hundred years, his theories are being applied in the study of black holes using the Ramanujan Strawman function. Pure mathematics is a curiosity which comes to you, a question which you would lack to find an answer to and those questions should be taken up, irrespective of whether it is used now or will be of any use to mankind or what good is it providing to the society.

Lakshmi: Do you think that scope for undergraduate research should be provided? How can students be encouraged in this direction?

Dr. Gupta: The research field is not always the same. It requires a lot of dedication, passion, and a lot of sacrifices. It is not always rewarding and involves a lot of patience. So, I will not encourage students to take up research or pure mathematics. But like the other side of a coin, if you think you can remain with a subject forever, you have passion for it, then definitely go for it. It takes a long time for anyone to get settled here and to do research. Some problems which we are working on don't get solved immediately and you'll need to stick with it; maybe when you are not able to solve that problem, there is some basic fundamental thing which you need first prove, which one is unable to see, so it will take time, it's not always very easy to get results. I love mathematics, I enjoy my books, I enjoy solving my problems, so if you get pleasure in doing so then only go for it.

Saloni: Mathematics is a subject that has a special beauty to it and you've mentioned how much you enjoy it. But what are the other activities you enjoy?

DR. GUPTA: Earlier before when I was a school student, I used to learn to draw and listen to good music, or I used to go places. But after coming to ISI the major thing I love is reading papers and doing research.

LAKSHMI: Now coming back to academia, especially the Zariski cancellation problem which quite made the landmark. Could you briefly explain and try to make the UG students understand what it is all about?

DR. GUPTA: It is about the cancellation. For example, if I say xa is equal to ya then you can cancel a from both sides and say that $x = y$ but this is not always true, for example, if you put $a = 0$ then you see $0 * 3 = 0 * 2$ but 3 is not equal to 2, so the point is that when we do cancel, what can we do is an important question. This is a very fundamental concern. We all know vector spaces; vector spaces are determined by their dimensions so if you have a vector space you add one more dimension to it and suppose it is isomorphic to another vector space plus one more dimension, then you know the original vector spaces were isomorphic because vector spaces are determined by their dimensions and so you can cancel vector spaces.

Zariski's cancellation problem is about the cancellation of affine spaces such as affine varieties, so it is not just a vector space but a vector space with a topology. In a very raw way of saying, suppose you have two different curves and you look at the cylinder over them, so over the curve, it will be like a surface and if these two surfaces are isomorphic, can you say the original surfaces were isomorphic? To have an affirmative answer for space of dimension less than equal to 2, what I have shown is that for spaces of dimension in positive characteristic greater than or equal to 4, they all have counterexamples so you cannot cancel in positive characteristics. This is all about the Zariski cancellation problem.

SALONI: Do you think mathematics is discovered or invented? For example, if we ever encounter an extraterrestrial civilization, do you think their math and our math will differ by symbols only? Or they'll have a whole different sense of the subject?

DR. GUPTA: I think it will probably lie around the same line but with differences. For example, earlier in the 6th or 7th century when ancient mathematics had developed, during the times of Aryabhatta, Brahmagupta, Bhaskara, our mathematics was at its peak. Thus, the topics like zero and decimal system went to Europe from India. The topics like Algebra, Polynomials, Continuity fractions, solutions of Pell's equation were defined at that time here. But when you search the internet for the same Pell's equation, it was a problem that was solved by Fermat in the 17th century. Basically, it took 1000 years for Europe to reach there. Unfortunately, they were quite fast and they didn't stop there and progressed high. On the other hand, in India there was decline in civilisation and we couldn't make up with the development. So, what I feel like is that if some extraterrestrial space comes up, they may be ahead of us or behind us but eventually we all will be somehow around the same path.

Lakshmi: How were you taught during your student years? Did your mentors have an impact on you? How has this changed your style of teaching your own students?

Dr. Gupta: Actually, it was my mother who taught me right from the beginning. In school, too, I was blessed with a very good set of teachers and so was the case in college. With this, fortunately, I landed up in ISI for my Masters and here also the teachers were very supportive. They introduced me to the whole world of Mathematics. During my transition period, when I was struggling to settle in from University to ISI, I was way behind of the curriculum, but the teachers were encouraging and they simply provided me with the best guidance to help me out by telling me the sources or the books to refer for the definitions and all. I, too, tried to maintain the same bond with my students from the past 10 years. I get very beautiful emails from them thanking me for my teaching, as they have succeeded now. And I believe that this is what I've learnt from my teachers only and I keep following their footsteps.

Saloni: What advice would you give to budding mathematicians concerning their academic pursuits and life in general?

Dr. Gupta: Mathematics is a very beautiful subject. Those who do it can actually feel that it's not comparable with any other subject. Even in Physics I used to enjoy it a lot but then there is a huge difference between what physics is and what mathematics is to me. Keep doing this. There is a high chance of having success in research and building a career in mathematics. There will be dry phases too, so try to bear with it. But if you enjoy it, you won't feel lonely. You must pursue this in the near future but not with the mere aim of earning money through this. You are the future of our country and you all must continue with research for the sake of our country.

Contents

120 IS NOT THAT SIMPLE

Adityendra Tiwari
Year III

Abstract
Taking away the 'Abstractness' from "*Abstract algebra*" and considering a simple example that gives a bird-view of what it incorporates is the main motive of this article. In addition, the article also tries to answer 'why' we are doing this in the first place.

Notations: (x_n) denotes sequence. $x_n^{(i)}$ denotes that *i* is the fixed index and n is the running index. $U \subset V$ includes both possibilities, $U = V$ and $U \subsetneq V$. $B(x, \gamma)$ denotes open ball centered at x with radius γ.

Introduction

Simple group theory lies at the heart of the study of groups itself. A notion that was first introduced by Évariste Galois about 190 years ago. Mathematicians have since been on a mission to classify all the simple groups, which is largely considered to be *completed* but the proof runs well over 10,000 pages and is spread over 100s of journals! The Simple groups act as building blocks in group theory and any 'finite' group can be decomposed in terms of factors consisting of simple groups that are independent of the choices of normal groups chosen at the time of factoring them. To keep the article to the point, not all the theorems are proved and some idea has been given as to why it has to be true. Undoubtedly 'the' most powerful tools in finite group theory have been used in this article and therein lies its beauty.

The Question:

Is a group of order 120 SIMPLE?

Tools:

Some terms that are required to understand the article and some results are in order:
We use Sylow's Theorems frequently in the answer to 'the question'. They basically say the

following:
1) (Existence) If the order of the group, say G, has factors consisting of primes and their powers then the group 'has' subgroups with order equal to the prime and their powers.
2) (Numbers) Number of p-SSG in a group of order, say, $p^k * m$ is equal to $(pk + 1)$ and this number divides m.
3) (Unique and Normal) If any p-SSG is unique in a group, then it is normal.

p-SSG: p-Sylow Subgroup, is the largest subgroup in G that consists of elements of order as power of p (prime) only.

SIMPLE GROUP: A group is said to be simple, if it does not contain any non-trivial proper normal subgroup.

NORMALIZER: $N(H) = \{x \in G \mid xHx^{-1} = H\}$

CENTRALIZER: $C(H) = \{x \in G \mid xhx^{-1} = h \text{ for all } h \in H\}$

RESULT: If the number of p-SSG in a group G is n_p then $[G : N(H)] = n_p$ where H is 'a' p-SSG.

N/C THEOREM: Let H be a subgroup of a group G. $\frac{N(H)}{C(H)}$ is isomorphic to a subgroup of $Aut(H)$, where $Aut(H)$ is the group of Automorphisms on H.

INDEX THEOREM: If G is a finite group and H is a proper subgroup of G such that $|G|$ does not divide $|G : H|!$, then H contains a nontrivial normal subgroup of G. In particular, G is not simple.

EMBEDDING THEOREM: If a finite non-Abelian simple group G has a subgroup of index n, then G is isomorphic to a subgroup of A_n.

Solution:

(Hypothesis) Let G be a 'simple' group of order 120. Then since $120 = 2^3 * 3 * 5$.
So, G contains 2-SSG, 3-SSG, and 5-SSG. (By Sylow's Theorem)
Following the notation used in the result, the possible values for n_p are:

$$n_2 = \{1, 3, 5, 15\}.$$
$$n_3 = \{1, 4, 10, 40\}.$$
$$n_5 = \{1, 6\}.$$

If any of the above number is one, then by the Sylow property it corresponds to a proper normal subgroup in G, i.e., G is not simple. So it cannot happen.
If n_2=3 then if H is a 2-SSG then by RESULT, $N(H)$ has index n_2, now, by the index theorem

G cannot be simple. Similar reasoning discards 4 for n_3. The Embedding Theorem comes to our rescue for n_2=5 since $|A_5| = 60$ and 120 does not divide 60.
Consider when $n_3 = 40$, then there are $40 * 2 = 80$ elements of order 3, $6 * 4 = 24$ elements of order 5. If H_1, H_2 are 2 distinct 2-SSG then $|H_1 \cap H_2|$ has at least 4 elements. So, there are $15 * 4 = 60$ more elements, i.e., in totality there are more than 120 elements! A contradiction.
This means n_2=15, n_3=10, n_5=6.
Now consider H to be a 3-SSG. Clearly, it has order 3, hence, it is isomorphic to $\mathbb{Z}_3$ so H is abelian. This means that $H \subset C(H)$. Now, by the N/C Theorem, $\left|\frac{N(H)}{C(H)}\right|$ = $|Aut(H)|$ = 2. This tells us that $|C(H)| = 12$ or 6. In either case $C(H)$ has an element of order 2 and with the help of the generator element in H this implies that G must have an element of order 6. Finally, the observation that n_5=6 says, if K is a 5-SSG then $N(K)$ has index 6, so, by Embedding Theorem, G is isomorphic to a subgroup of $\mathbb{A}_6$. But $\mathbb{A}_6$ does not have an element of order 6!

This tells us that 'any' group of order 120 cannot be simple. □

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BEAUTIES THAT MAKE π UNIQUE

Dushyant Kumar Rohilla
Year II

Abstract
Pi(π) appears everywhere in the universe and every time in our lives. It is literally woven into our universe; the orbits of planets, the electromagnetic waves, rivers, the colours of auroras, the structure of DNA, the Great Pyramid of Giza, and so on. Pi(π) helps us to see the mathematical ideas underlying diverse physical processes. In this paper, we will witness some fascinating beauties about Pi(π).

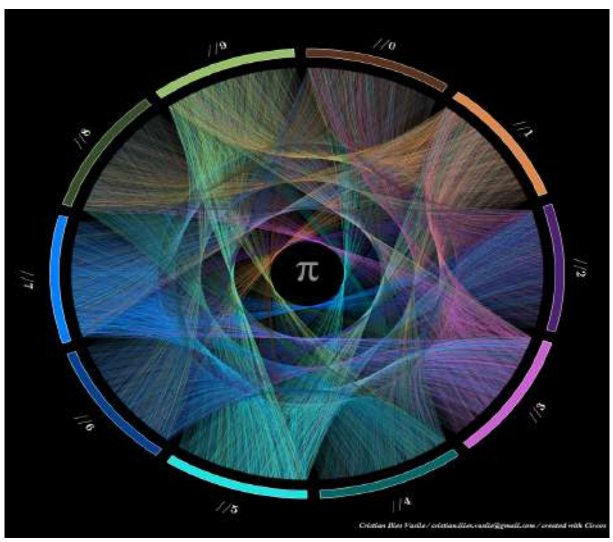

Figure 2.1: An infographic created using first 1000 digits of π.

Introduction

Pi(π), what is it? The most famous mathematical constant, π, is the ratio of a circle's circumference to its diameter. It is usually approximated as 3.14159. Suppose that, if I have a bicycle tyre with a diameter of 1, then one complete revolution of the bicycle tyre will cover a distance of π.

Pi(π), the number which we obtain on dividing the circumference of a circle to its diameter — 3.1415926535 — is just the beginning. It keeps going forever without ever repeating. It means that contained within this string of decimals might be every other number; your birthdate, the combination to your locker, your social security number, etc. It all can be in there somewhere. And if you map these decimals with letters, you would probably have each word that ever existed in every possible combination; the first syllable you spoke as an infant, the name of your latest crush, your complete life story from starting to finish, and everything we ever say or do. All the world's infinite possibilities are possibly present within this one simple circle!

Is π "Normal"?

Unfortunately, mathematicians have not proved yet that π has the characteristic of "normality". Alternatively, mathematicians are not certain if π contains all the finitely long permutations of digits from 0 to 9. They are not certain if every digit is used after a certain amount of time or an infinite number of times in π's decimal representation.

When we check the first billion digits of π, we find that the digit 7 occurs almost 100 million times. This makes π a superb random number generator because the probability of getting the digit 7 should be $\frac{1}{10}$ or 10%. However, after some points, π might not contain the digit 7 and may have a non-recurring number with only two or three digits, such as 0102031122330001112223333....

For instance, after the first 761 digits of π, there is a famous mathematical coincidence where six nines occur consecutively, which is known as the **Feynman point**.

But we are sure that the digits of π keep going on forever and in random order. This randomness makes π interesting because π's value is finite; however, its decimal value is infinitely long. This might seem like a contradiction, however it's not. The number π is a constant number because it is the ratio of a circle's circumference to its diameter, which are finite values.

If π is a normal number, then we can say that our whole destiny is encoded in π. The pictures we will take in the future will be in π because there are binary numbers behind images. All digital products are in π. Even this paper might have been in π for thousands of years. Furthermore, the DNA of every creature might be in π. All this might be true, but we are not certain yet!

The Connection Between π And Sinuosity Of Rivers

Pi(π) features a direct connection with rivers on Earth. To figure out this relationship, we are required to measure the length of a river in two different ways. Assume that we know the initial and the ending point of the river. First, we need the actual length of the river. In other words, the actual length is the distance that you need to swim from the beginning point to the ending point of this river. Let us denote this length by “L”. Second, we need to find the straight length, i.e., this time we need to find that distance which we would cover if we fly from the beginning to the end point of the river. Let this length be “l”. Now we can write the formula for the sinuosity of the river, which is just the ratio of L to l. The sinuosity measures how bendy the river is.

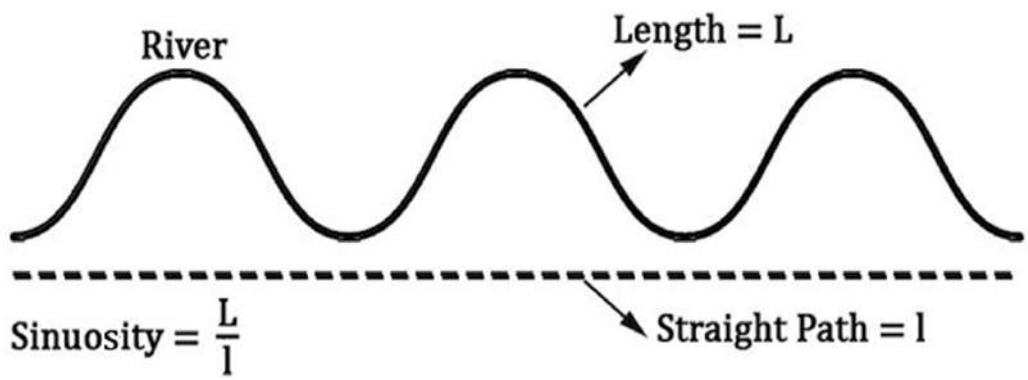

Figure 2.2: The ratio is often found to converge to (but rarely exceed) 3.14, roughly π.

In 1996, a scientist named Hans-Henrik Stølum came up with a theory that the average sinuosity of rivers around the world is π. If you find all the rivers’ sinuosity and take the average sinuosity, you would get π.

There’s another fascinating fact about sinuosity. Rivers can be very bendy at certain points. So, we would expect a high sinuosity. But, those rivers become straight and make the sinuosity closer to π. This phenomenon is called an oxbow lake. So, it is hard to find a river’s sinuosity equal to 7 because of fluid dynamics. Mathematicians found the highest sinuosity of rivers to be around 3.5 and the lowest sinuosity to be around 2.7.

π In Space

Our universe inherits a mathematical order. For instance, to know our solar system, we need π. We know that our planets revolve around their host stars and the light comes from those host stars. To talk about that light, we need to know how massive the host star is. In other words, we require the surface area of the host star. The formula for a sphere's surface area is $4\pi r^2$, with r being the star's radius. The size of a planet conjointly helps scientists to guess whether it is habitable or not.

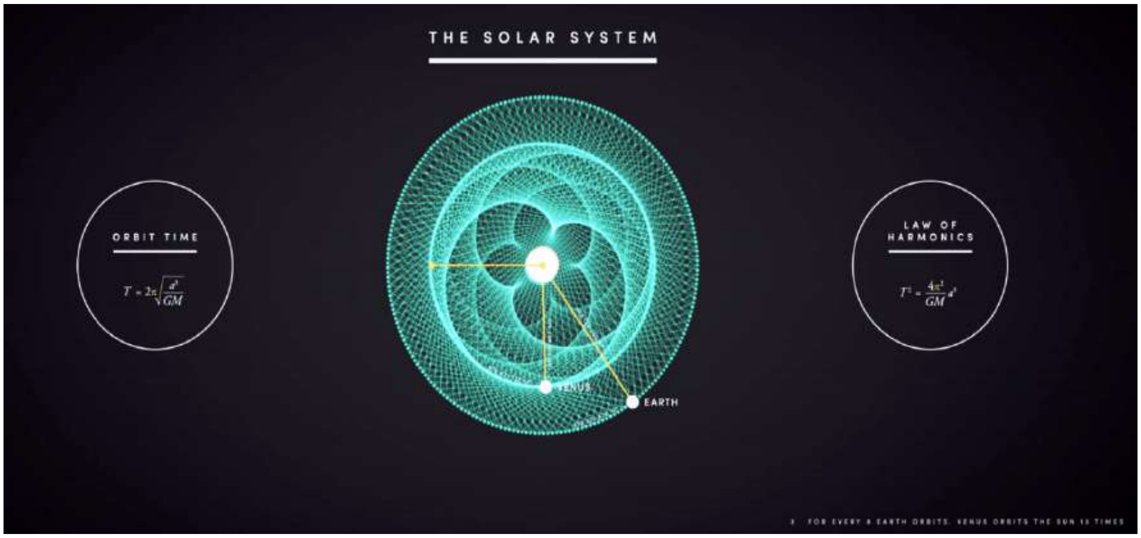

Figure 2.3: For every 8 Earth orbits, Venus orbits the Sun 13 times.

Another example to show the connection between π and the universe is the electrostatic force, i.e. the force acting between two electric charges. An electron exerts a force in every direction and forms a spherical field. Electrons also interact with each other due to their electric fields. To figure out that interaction, we need to find the surface area of spheres, where again π shows up.

There is also a connection between π and gravity. In Einstein's field equations, you might notice that π is there too.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where the Ricci tensor R and the metric tensor g depict the structure of spacetime, the stress-energy tensor T depicts the energy and momentum density and flux of the matter in that point in spacetime, and the constants G and c, the universal constant of gravitation and the speed of light, are conversion factors that emerge when utilizing traditional units of measurement. When Λ is zero, this reduces to the field equation of general relativity typically used in the 20th century. When T is zero, the field equation portrays empty space (the vacuum).

So π is the part of gravity, energy and momentum of the universe and all objects which exist within it. You won't find any other irrational number to be involved that much in our universe!

CONCLUSION

Mathematics is a language that is inscribed within the brains of all human beings and π is just a word in that language. Perhaps one day the great mathematicians (maybe you) will reveal all the mysteries hiding behind the digits of π.

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UNDERSTANDING MATHEMATICAL KNOTS

Khyati Khera
Year II

Abstract
Anything authentic goes through its share of legendary discoveries, and so did the fascinating knot theory. In this article, we introduce the mathematical knots and the way one knot is distinguished from another.

Keywords: *Knot, Reidemeister moves, Equivalent Knots, Tame and Wild Knots, Framed Knot, Harmonic Knot, Ambient Isotopy, Reidemeister's theorem*

Introduction

Knot theory, as its name suggests, is the study of mathematical knots. It studies the closed curves in three dimensions, and their possible deformations without one part cutting through another. Unlike the common perception of a knot, such as that of a rope or a shoe lace knot, a mathematical knot has no loose ends which rules out the possibility to undo the existing knot and tie it in a different manner.

The conjecture by William "Lord Kelvin" Thomson in the 1860s that atoms were knots led to the first extensive study of knots. Although his ideas were thereafter proved to be false, his theory inspired many mathematicians to follow up and formulate the modern day theory.

Definition

A mathematical knot is defined as an embedding of the circle S^1 into the three-dimensional Euclidean space, $\mathbb{R}^3$. [1]

A knot is a closed loop of a string where the string has no thickness at all. It can't cross over itself because that would result in branching in the string. These crosses that the string makes over itself are called **crossings** of a knot.

Definition

A crossing is a location on a projection of a knot where two strands appear to cross. The strand that can be seen completely in the crossing is called the overstrand and the one that is obscured by the overstrand is called the understrand. [5]

BASIC MATHEMATICAL KNOTS

Mathematicians tend to name all the knots in a way that tells us the basic information personal to that particular knot. A knot named x_y has two parts; x shows us the ***crossing number***, that is the minimum number of crossings in the knot diagram of that knot and y shows us the ***knot index***, which is an arbitrary index number assigned to a specific knot of same crossing number.

Let's identify a few of the common knots in the given figure 3.1.

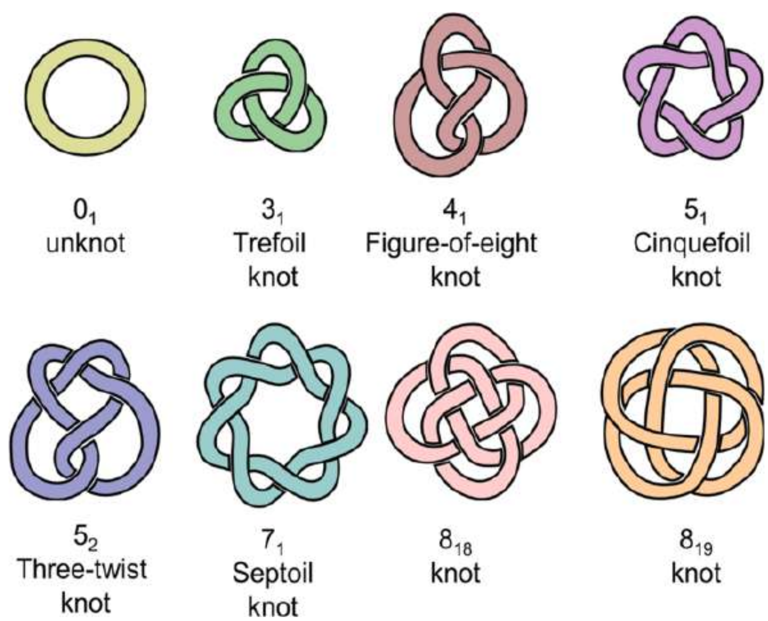

Figure 3.1: Some Mathematical Knots

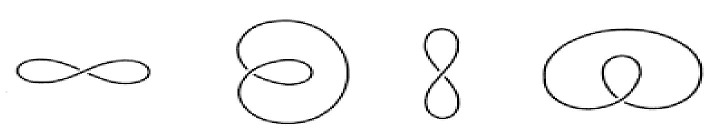

Figure 3.2: One Crossing Knots

The Unknot or The Trivial Knot- 0_1

The unknot has 0 crossings, that is, it is a closed loop without any knots. Figure 3.2 shows four one crossing knots that can be transformed into the unknot without cutting it or passing it through itself. Hence, even though these knots look different to the unknot, they are, in reality, not. These are called the ***projections*** of the unknot. Therefore, there is only one existing knot with zero crossings, making the knot index 1.

Trefoil Knot- 3_1

The trefoil knot is a knot with 3 crossings and 1 as its knot index. It is the simplest example of a non-trivial knot.

Figure-of-eight Knot- 4_1

This is the most commonly used knot in fundamentals of knot theory. It has 4 crossings and knot index 1 and its name is derived from the fact that its knot diagram gives the appearance of the digit 8.

Types Of Knots

Tame versus Wild Knots

A tame knot is a knot equivalent to a *polygonal knot*; a knot whose image in $\mathbb{R}^3$ represents a finite set of line segments joined together. On the other hand, a wild knot is the opposite of the tame knot, which shows deviant and irregular properties.
The unknot, trefoil and figure-of-eight are tame knots. Given below is an example of a wild knot.

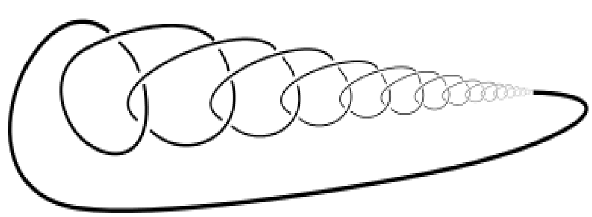

Figure 3.3: A wild knot

Framed Knot

Think of a framed knot as an embedded ribbon with framing as the number of twists performed around the knot. Formally, a framed knot is a tamed knot with its closed-curve solid $D^2 \times S^1$ in S^3 where the D is the disk and S is the surface.

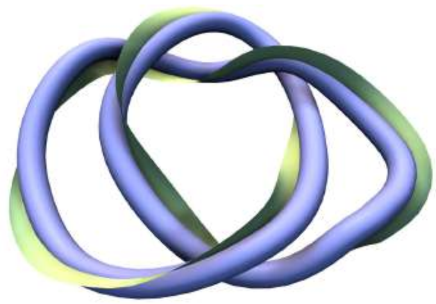

Figure 3.4: A framed knot

Harmonic Knots

Knots represented in parametric form are called harmonic knots; in the form (x(t), y(t), z(t)) where each coordinate is a function of a trigonometric polynomial.

REIDEMEISTER MOVES

But, there is an impending question at this point which requires us to know if the trefoil and the unknot are indeed two different knots and cannot be transformed via ambient isotopy.

Definition

In the mathematical subject of topology, an ambient isotopy, also called an h-isotopy, is a kind of continuous distortion of an ambient space. For example in knot theory, one considers two knots the same if one can distort one knot into the other without breaking it. [3]

Two mathematical knots are equivalent if one can be transformed into the other via a deformation of $\mathbb{R}^3$ upon itself, known as an ambient isotopy. These transformations correspond to knotted string manipulations that do not involve cutting or passing through it. More precisely, $f_1 : S^1 \hookrightarrow \mathbb{R}^3$ and f_2: $S^1 \hookrightarrow \mathbb{R}^3$ are equivalent if and only if there is an orientation preserving homeomorphism of pairs ($\mathbb{R}^3$,f_1 (S^1)) $\hookrightarrow$ ($\mathbb{R}^3$,f_2 (S^2)). [2]
Equivalent knots are, thus, regarded to be of the same type. A ***knot type*** is an equivalence class of knots. For instance, a knot equivalent to the circle $x^2 + y^2 = 1$, $z = 0$ is called a

trivial knot and accounts for the trivial type.

For several years, many mathematicians have tried to come up with a method that lets us know if two knots are ambient isotopes. Finally, in the 1930s, Kurt Reidemeister established the existence of knots other than the unknot by proving that they exist. He accomplished this by demonstrating that all knot deformations may be reduced to three sorts of “moves”; namely

- Twist, (and untwist in either direction)
- Poke, (move one loop completely over the other) and
- Slide (move a string completely over or under a crossing).

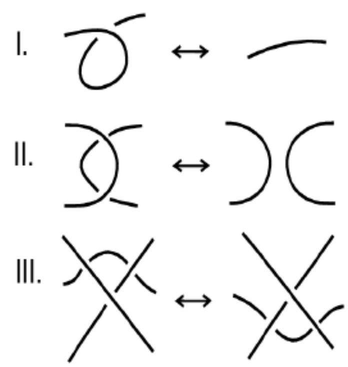

Figure 3.5: Reidemeister Moves

These are called the Reidemeister moves.

Definition

All the knots that can be reduced or deformed into an unknot via the Reidemeister moves are called culprit knots.

Reidemeister's Theorem

Two link diagrams represent the same ambient isotopy class of a link in S3 if and only if they are related by a finite number of Reidemeister moves. [4]

Where, in knot theory, a link is one or more disjointly embedded circles in three-space. In other words, a link is a collection of knots, entangled with each other.

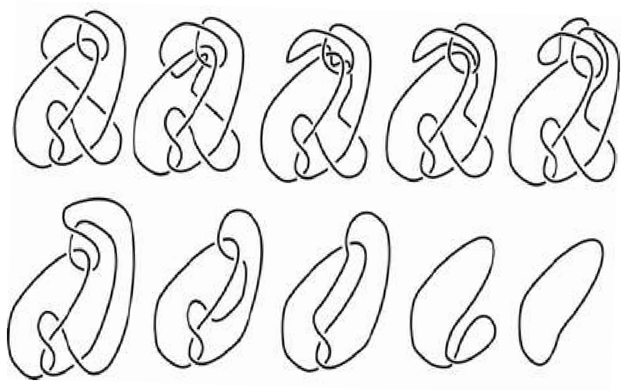

Figure 3.6: Culprit Knot

Definition

If the diagrams of links are related to each other only by moves of type II and III, then they are said to be regular isotopic. [4]

CONCLUSION

With this, we have covered the fundamentals of mathematical knots and it goes without saying that there is a lot, still left to be learnt. Mathematical knots are as interesting as you let them be, because the possibilities to this particular field are unlimited and once you delve in, you are engrossed for a lifetime. Knot Theory involves knot invariants, links and so much more and its application transcends through major disciplines like Biology; where it is used to untangle DNA, Cryptology and GPS applications.

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HILBERT'S HOTEL AND INFINITY ∞

Shubham Suman
Year II

Abstract
The purpose of this article is to provide a brief description of the **'Hilbert's Paradox of Grand Hotel'**.It incorporates the concept of transfinite or ordinal numbers (sometimes called **ordinals**). This is a paradox that deals with infinite set comparisons and is adapted and extended to include ideas from calculus, specifically infinite series. The actual demonstration of this paradox is a hotel with an infinite number of rooms that are always full but can still accommodate a finite or infinite number of guests and can repeat the entire process.

Keywords: *Ordinals, Sequence, one-to-one correspondence, Countable infinity, Uncountable infinity*

Introduction

David Hilbert (January 23, 1862 – February 14, 1943) was a German mathematician who was one of the most influential mathematicians of the nineteenth and early twentieth centuries. Hilbert discovered and developed a wide range of fundamental ideas in a variety of fields, including invariant theory, calculus of variations, commutative algebra, algebraic number theory, the foundations of geometry, spectral theory of operators and its applications to integral equations, mathematical physics, and the foundations of mathematics (particularly proof theory).[1] He used to use stories about infinitely many rooms to illustrate his popular lectures, demonstrating how an actual infinite is not an absurd idea.

Hilbert's Paradox Of Grand Hotel

There is a grand hotel comprising an infinite number of rooms. Even though this Grand Hotel is always fully occupied, it is universally known as never refusing any new guest who asks for a room at any time.[2]
Assume we assign a number to each room in the hotel. To begin, we also require a first

ordinal, denoted by the number 0. (Strictly speaking, the second principle for generating ordinals gives us 0, since zero is the First ordinal after the empty sequence.) [3]
Assume we assign numbers to rooms as follows: Room 0; Room 1; Room 2; Room 3; Room 4; Room 5; Room 6; Room 7; Room 8; Room 9; ......Room n and so on. All the rooms are full of guests.

One New Guest

If a new guest arrives at the hotel and requests a room, the receptionist will never turn him away and will always assign him a new room, even if the hotel is always fully occupied. What makes this possible?
The solution is straightforward!
Suppose, a new guest G comes and asks for a room. Since all the rooms are filled with infinitely many guests, where will we put the guest G? The answer is simple! We shift the guest in the 0^{th} room to the 1^{st} room, the guest in the 1^{st} room to the 2^{nd} room, the guest in the 2^{nd} room to the 3^{rd} room, the guest in the n^{th} room to the $(n+1)^{th}$ room and so on.

Now, Room 0 is free and guest B can occupy room 0. Here, we can see that we can find one-to-one correspondence with set A (infinitely many rooms) to the Set containing $\omega + 1$ elements (ω infinitely many guests + one extra guest).

Finitely Many Guests

We can do this process for k finitely many guests by shifting the guests in room n to room $(n+k)$. Thus, the hotel can still accommodate k finitely many guests even if the rooms are occupied by the previous guests. We can still find one-to-one correspondence even if we have finitely many new guests, i.e. we can map one-to-one correspondence between Set A and Set containing $(\omega + k)$ elements (ω infinitely many guests + k extra guests).

Infinitely Many Guests

But what if we have infinitely many guests? The receptionist doesn't have a problem with that either. He can allot rooms to infinitely many guests without being the condition in which two persons are in the same room. He will simply shift the person in room 1 to room 2, guest in room 2 to room 4, guest in room 3 to room 6, guest in room n to room $2n$ and so on. Now we have a situation in which there are an **infinite number of odd-numbered rooms available** and we can accommodate an infinite number of guests. Here, it looks absurd but we can still find one-to-one correspondence between two sets, set A and set containing $(\omega + \omega = 2\omega)$elements ($\omega$ infinitely many already existing guests + ω infinitely many new guests).

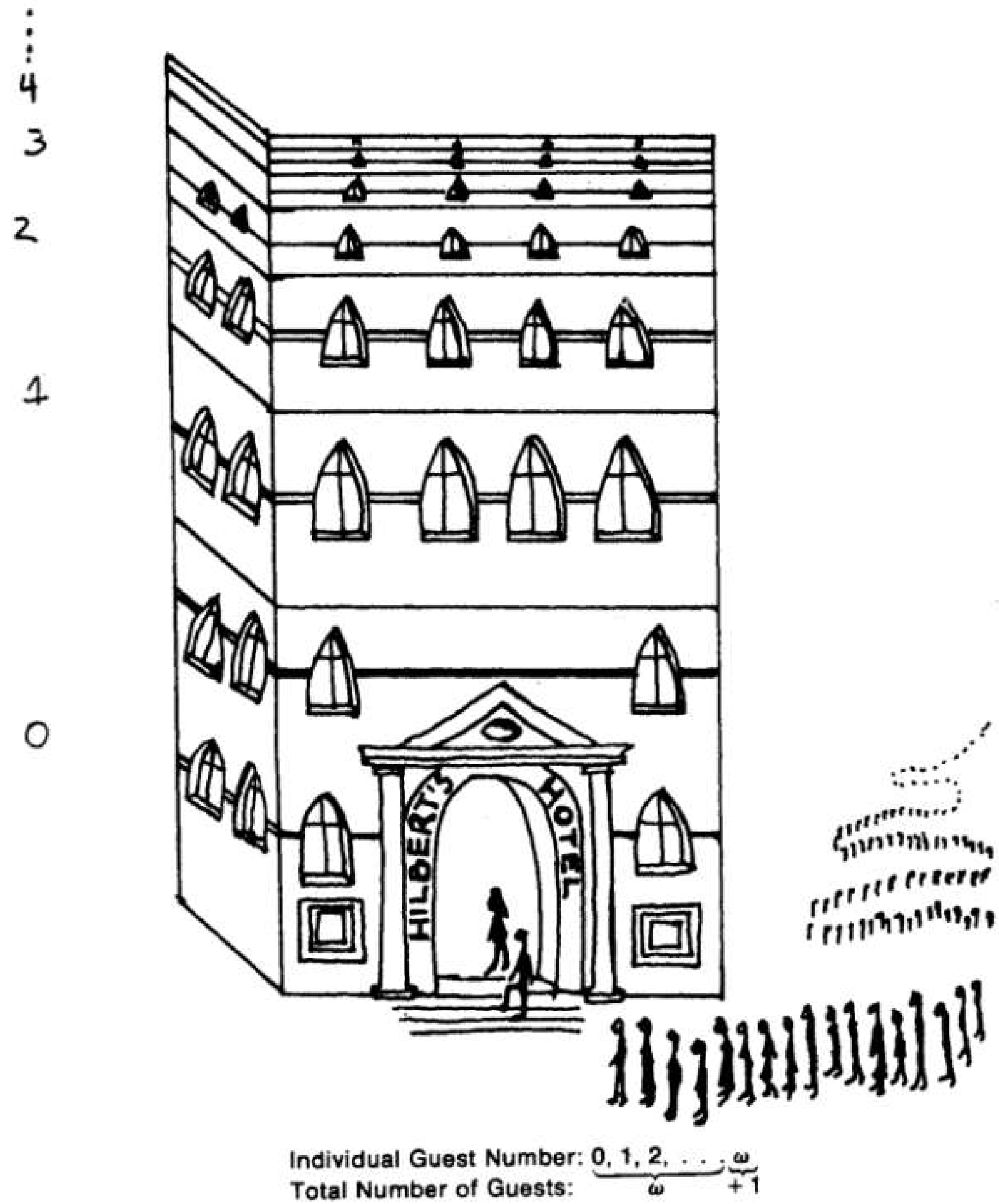

Figure 4.1: Hilbert's Hotel [3]

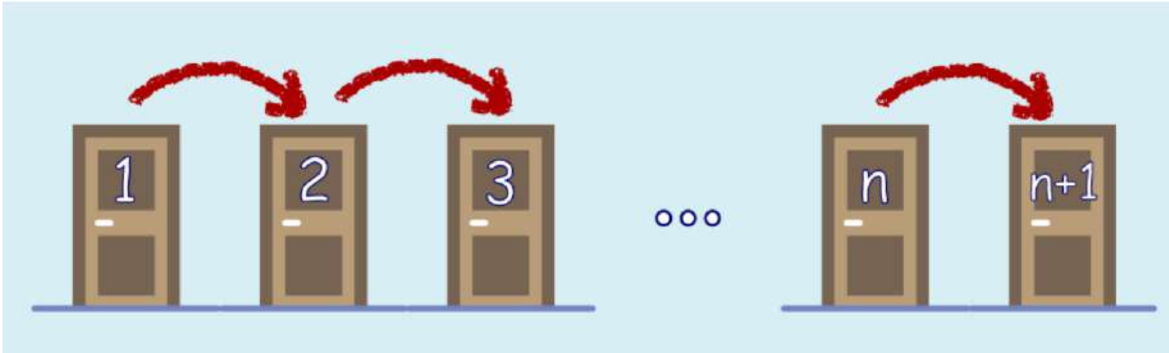

Figure 4.2: Demonstration of Shifting of Guests [7]

Doubling The Layer Of Infinity

Suppose an infinite number of buses come, containing infinitely many guests. Then, can we still say that we can accommodate those infinitely many guests too? The answer is still YES!

Suppose we number the buses as 1, 2, 3, and 4 ...so on. Also, we number the guests seating on the seats of the buses as 1, 2, 3, and 4 ...so on. Then we ask the existing guests in the hotel to shift to the rooms whose number is twice the number of the room in which they're staying. i.e. the guest in room 1 to room 2, room 2 to room 4, room 3 to room 6 and so on. Now we have **infinitely many odd-numbered rooms free**. Then, we ask the guest in the 1^{st} bus on the 1^{st} seat to move to room $3^1 = 3$, the person in the 1^{st} bus on the 2^{nd} seat to room $3^2 = 9$, the person in the 1^{st} bus on the 3^{rd} seat to room $3^3 = 27$ and so on.
We follow the similar pattern for guests in the 2^{nd} bus. We ask the person in the 2^{nd} bus on the 1^{st} seat to move to room $5^1 = 5$, the individual in the 2^{nd} bus on the 2^{nd} seat to move to room $5^2 = 25$, the guest in the 2^{nd} bus on the 3^{rd} seat to move to room $5^3 = 125$ and so on. The same follows for the guests in bus 3. We ask the person on the 1^{st} seat to move to room $7^1 = 7$, the guest on the 2^{nd} seat to move to room $7^2 = 49$, the guest on the 3^{rd} seat to move to room $7^3 = 343$ and so on. In fact, the guests in bus n can be adjusted in the hotel by allotting the room p^k to the person on seat k, where p is the $(n+1)^{th}$ prime number. The questions which arise, then, are a.) whether all these rooms are really free? b.) Isn't it also feasible for two people to be assigned to the same room?

The answer to the first question is YES! And the answer to the second question is NO. The justification of the answer to both the questions is given by "Fundamental Theorem of Arithmetic", which states that every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.[4] So, suppose a number z is a room number which is a power of a prime number, it can't be a power of another prime number. For example, the room allotted to the guest in bus number 3 on seat 3 will be room $7^3 = 343$, now if we want to allot the same room to the guest in bus 2 on seat 4, then 343 would be $5^4 = 343$ which is not the case as $5^4 = 125$ and not true according to the Fundamental Theorem of Arithmetic.

As a result, each passenger on an endless number of buses carrying an infinite number of passengers is assigned a room without regard to whether or not two people are in the same room. Now, we have mapped one-to-one correspondence between Set A and Set containing ω^2 elements.

TRIPLING THE LAYER OF INFINITY

Assume that Hilbert's large hotel is located on the coast. Assume that an infinite number of ships arrive, each containing an infinite number of buses, each with an infinite number of passengers. We now have three layers of infinity: an unlimited number of ships, an infinite number of buses, and an endless number of guests.
Is it still possible to argue that the hotel can accommodate such triple layers of infinitely many guests?

Surprisingly, the answer is still YES!

We'll follow the same pattern again. Suppose the guest in ship s on bus b on seat r wants a room. We'll first find $(b+1)^{th}$ prime number, let's call it p then, this p is raised to power of seat number r which gives p^r. Now, we'll find $(s+1)^{th}$ prime number and let's call it q. So, we'll allot the guest in ship s on bus b on seat r the room, whose room number is equal to

$$R = q^{p^r}$$

where R corresponds to the room number. For example, the guest in ship 2, in bus 1 on seat 2 will be allotted the room whose room number is equal to

$$\begin{aligned} \text{Room Number(R)} &= 5^{3^2} \\ &= 5^9 \\ &= 1,953,125 \end{aligned}$$

The guest in ship 3, in bus 2 on seat 3 will be allotted a room whose room number is equal to

$$\begin{aligned} \text{Room Number(R)} &= 7^{5^3} \\ &= 7^{125} \\ &= 4337654948097993282537354757263188251697832994 \times 10^{60} \end{aligned}$$

As time goes on, the numbers will grow larger. However, we can accommodate triple layers of visitors, as we just saw. Isn't it also possible for two people to be allocated to the same room, as well? The answer is no, and it is supported by the Fundamental Theorem of Arithmetic.
Another issue whose answer is interesting to know is how far into infinity can we go and the accommodating people's answer remains yes.

Infinitely Many Layers Of Infinity

The answer is that we can go to a finite number of levels of infinity but not an infinite number of layers. The answer to which remains unknown and unfathomable. That is, we can find or map one-to-one correspondence between Set A and Set containing ω^k elements, where k is a finite natural number but not between Set A and Set containing ω^ω elements. We'll prove this by showing that if we're able to allot rooms to guests from infinity raised to power infinity, then we'll find one such guest who doesn't have a room of his/her. Then we can't say that there exists a one-to-one correspondence between the set A (infinitely many rooms) to Set ω (guest numbers equal to infinity raised to power infinity). If we say that we can still allot rooms to guests from infinitely many layers of infinity guests,

$$\text{Guests} = \omega^\omega$$

then let the rooms allotted are numbered as 0, 1, 2, 3 ...And so on. Now, how will we identify each guest? We will need a system to identify each guest and to allot the rooms to each guest. We'll be using the word **'identity'** for the belonging of a guest from seat s , in bus b, in ship k, in ocean o, on planet p, on star l, in galaxy g, in supercluster c and so on.

Now, our aim to identify a single guest who doesn't have a room of their own.

Suppose, we go to room 0 and ask the guest in this room, "What's his seat number from his identity?" The guest says that he's from seat 1, of bus 1, in ship 1, in ocean 1, on planet 1, on star 1, in galaxy 1, in supercluster 1 and so on.
Now, the seat number of guest in room 0 is 1. What we're trying here is to achieve an identity of a guest, and later we'll prove he doesn't have a room. So, the identity of this new person will be seat 2 (any number except 1), in bus, in ship, in ocean ..., and so on.

Now, we'll go to room 1 and ask the guest in room 1, "What's her bus number from her identity?" The guest says she's from bus 2.
The identity of the guest which we're finding will be seat 2(any number except 1, bus 3(any number except 2) in ship, in ocean ...and so on. We move to the next room, and repeat the process, and keep going for every room of the hotel.
The illustration of this process will look like this

$$\begin{aligned} I_0 &= s1, b1, \ldots \\ I_1 &= s2, b2, \ldots \\ &\vdots \\ I_n &= s(n), b(n), \ldots \\ &\vdots \end{aligned}$$

The new identity will be

$$I_r \quad = \quad s2, b3, \ldots$$

where I_r denotes the new identity.
Now, what we're building is that we're finding **an identity of a guest which doesn't belong to any guest already residing in the hotel**, thus, proving the assumption wrong.
Suppose by repeating the process and by going to every room of the hotel, we generate a **new identity**. This identity is called new because we have countably infinite options for choosing numbers for seat, bus, ship, ocean ..., and so on. This **new identity** certainly doesn't belong to any guest already residing in the hotel. Thus, our assumption is proved wrong here.
Suppose any guest in the hotel claims to have the identity which we created, but it simply can't because we're here repeating the process of asking every guest already in the hotel. So, any guest who is already residing in the hotel can't have the identity which we created. That's why our assumption is wrong here.
Hence, we'll have a guest's identity which doesn't belong to any of the guests in the hotel and thus confirming that we can't accommodate infinitely many layers of infinity guests in Hilbert's Grand Hotel.
What did we get by this Hilbert's hotel consisting infinite number of rooms which are already occupied by the guests?
When we are going to talk about infinities, the basic arithmetic rules fail; they don't apply. For example, $\infty + k = \infty$ where k is any real number.

Also, this illustration which is given by David Hilbert illustrates that countable infinity raised to the power of countable infinity is uncountable infinity. It is larger in some fundamental sense than countable infinity.

The reason behind this statement can be explained by follows: What seat number in identity of a guest can we generate for guest who doesn't have any room? The answer is countable infinity minus 1, which is still countable infinity. The same follows for the bus number, ship number till countable infinity. So, we have the number of guests not in the hotel as countable infinity times countable infinity times countable infinity ..., countably infinite times. In other words, no matter what system or process we employ to allot rooms to guests, there are countable infinity raised to the power of countable infinity guests who are not allotted rooms in the hotel.

This proves that uncountable infinity is uncountably larger than countable infinity. Thus, we know that infinity raised to the power of infinity is uncountably larger than our countable infinity.

CONCLUSION

The paradox here is that even if the hotel is always full, it can still accommodate an infinite number of guests and repeat the entire process indefinitely. The illustration here is that absolute infinity is a possibility rather than a precise number. If infinity is a number, then Hilbert's grand hotel will never exist, and the rooms will eventually be filled with guests. The aim of this thought experiment was to demonstrate that "in the world Of infinity a part may be equal to the whole" [5] and that the notion of an actual infinity is possible and does not lead to absurdities, however counterintuitive it may be.[2]

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THE FAMILY OF METALLIC RATIOS

Sanskriti Jha and Khushi Agarwal
Year II

Abstract
This paper attempts to familiarize the readers with the family of metallic ratios. Initially, we have given an overview to the known Golden Ratio. Then we have discussed the Silver Ratio, followed by the generalization of the Metallic Ratio in terms of a quadratic formula. We have tried to establish some relationship between the Metallic Ratio and the famous Fibonacci sequence.
The paper also provides you some glimpse of the visuals to the Golden and Silver rectangles and spirals. All the above properties have been generalized for the different metallic ratios, but we have also discussed a property, the metallic polygon, which hasn't been generalized yet. These metallic polygons have been observed only in the case of golden and silver ratios till date. Research is yet to be done for other ratios.

Introduction

Although a large amount of literature has been written and published about the Golden Ratio, few people are aware of its generalized version known as Metallic Ratios, which is introduced in this paper. The methods for obtaining them were also thoroughly discussed. This will aid in further exploration of the universe of real numbers. Sequences play an important role in understanding the complexities of any given problem that consists of some patterns in mathematics.

The Golden Ratio

Before getting to the family of metallic ratios, let us give you a brief idea of the familiar Golden Ratio.
Consider a line segment as shown in the figure;
Let us divide this line segment in two divisions, A and B, such that $A > B$ and the ratio $A : B = (A + B) : A$
This ratio is defined as the "Golden Ratio" and is denoted by the Greek letter ϕ.
We can calculate its numerical value by expressing it mathematically as:

Figure 5.1: A Golden Line

$$\phi = \frac{A}{B} = \frac{A+B}{A}$$
$$\phi = 1 + \frac{B}{A}$$
$$\phi = 1 + \frac{1}{\phi}$$
$$\phi^2 = \phi + 1$$

The above quadratic equation, when solved using the quadratic formula, yields the solutions;

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

Since the length cannot be a negative quantity, we neglect the negative solution, which leaves us with only one solution, i.e.,

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.618033$$

The Silver Ratio

Now, instead of dividing the line segment into two divisions, let us divide it into three divisions such that the larger two segments are equal in length as shown;

Figure 5.2: A Silver Line

The above line segment is divided such that the ratio $A : B = (2A + B) : A$. This ratio is defined as the “Silver Ratio”, denoted by the Greek letter σ.

We can derive its value mathematically as follows;

$$\sigma = \frac{A}{B} = \frac{2A+B}{A}$$
$$\sigma = 2 + \frac{B}{A}$$

$$\sigma = 2 + \frac{1}{\sigma}$$
$$\sigma^2 = 2\sigma + 1$$

This again leaves us with two solutions;

$$\sigma = \frac{1 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

The negative one can be ignored, leaving us only with the positive solution,

$$\sigma = 1 + \sqrt{2} = 2.414213...$$

As observed, the Golden Ratio is obtained by dividing the line segment into two parts, whereas the Silver Ratio is obtained by dividing it into three parts.
Similarly, we can generalize the existence of other metallic ratios for further number of divisions of the line segment.

The Generalized Formula for Metallic Ratios

Suppose, we divide the line segment into n divisions of equal length A and a smaller division of length B such that the ratio $A : B = (nA + B) : A$. Then, the metallic ratio ρ_n, for the different values of n is given by,

$$\rho_n = \frac{A}{B} = \frac{nA + B}{A}$$
$$\rho_n = n + \frac{B}{A}$$
$$\rho_n = n + \frac{1}{\rho_n}$$
$${\rho_n}^2 = n\rho_n + 1$$

The solutions for this equation are:

$$\rho_n = \frac{n \pm \sqrt{n^2 + 4}}{2}$$

Ignoring the negative value, we find the value of the ratio to be,

$$\rho_n = \frac{n + \sqrt{n^2 + 4}}{2}$$

We can obtain different metallic ratios for different values of n, n >1, for example, the Golden Ratio for n = 1 and the Silver Ratio for n = 2.
An interesting point to be noticed here is that the conjugate of ρ_n obtained above turns out to be the reciprocal of ρ_n.
Another interesting property shared by all the metallic ratios is their representation in the form of Continued Fractions.
This can be mathematically written as,

$$\rho_n = n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \ldots}}}}$$

$$\rho_n = n + \frac{1}{\rho_n}$$

Relation with Fibonacci and other Sequences

Focusing our discussion on the Golden Ratio initially, we try to explain the relation of other ratios too with their corresponding sequences, respectively.

We have already seen the equation for Golden ratio:

$$\phi^2 = \phi + 1$$

On multiplying the equation by ϕ^{n-2} on both sides, n $\geq$ 2, we obtain,

$$\phi^n = \phi^{n-1} + \phi^{n-2}$$

...(1)

Now consider the sequence,

$$1, \phi, \phi^2, \phi^3, \phi^4, \phi^5 \ldots$$

...(2)

According to equation (1), each term of this sequence is the sum of the first two terms which is quite similar to the Fibonacci Sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ...$$

In the sequence (2), the ratio of the n^{th} term to the $(n-1)^{th}$ term is equal to ϕ, which is not the case with the Fibonacci Sequence. However, in the Fibonacci Sequence, this ratio converges to ϕ as the sequence progresses, as shown below.

$$1, 2, 1.5, 1.6666, 1.6, 1.625, 1.6154, 1.619, 1.6176, 1.6181, 1.618....$$

Now, taking forward this discussion to the Silver Ratio, we see that

$$\sigma^2 = 2\sigma + 1$$

$$\sigma^n = 2\sigma^{n-1} + \sigma^{n-2}, n \geq 2$$

Considering the sequence $1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, ...$ we see that the n^{th} term of the sequence is equal to the sum of twice the $(n-1)^{th}$ term and the $(n-2)^{th}$ term, which is similar to the Pell Sequence:

$$0, 1, 2, 5, 12, 29, 70, 169, ...$$

In the Pell Sequence, the ratio of the consecutive terms converges to the silver ratio as the sequence progresses.

Similar results hold for all other metallic ratios where the m^{th} term of the sequence

$$1, \rho_n, \rho_n^2, \rho_n^3, \rho_n^4, \rho_n^5...$$

is equal to the sum of n times the $(m-1)^{th}$ term and the $(m-2)^{th}$ term for all values of $m \geq 2$.

METALLIC RECTANGLES AND SPIRALS

Golden Rectangle

A golden rectangle is a rectangle whose sides are in the golden ratio. If you form a square on the smaller side of a golden rectangle and remove it, then the rectangle left is also a golden rectangle. This way you can create infinite number of golden rectangles which get smaller and smaller in size. If one draws quarter circles in these successive squares formed, a golden spiral can be obtained.

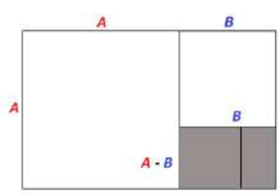

Figure 5.3: A Golden Rectangle

Silver Rectangle

A silver rectangle is a rectangle whose sides are in the silver ratio. If you form a square on the smaller side of this rectangle and then form another square adjacent to it, then the smaller rectangle left is also a silver rectangle. Similarly, you can obtain an infinite number

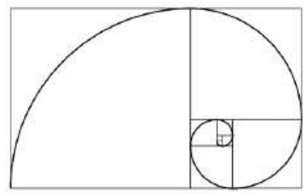

Figure 5.4: A Golden Spiral

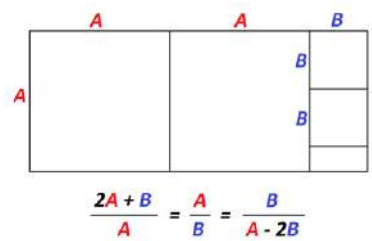

Figure 5.5: A Silver Rectangle

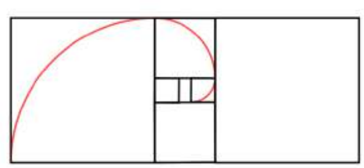

Figure 5.6: A Silver Spiral

of silver rectangles, each smaller than the previous one. If we rearrange these squares and rectangles and construct quarter circles as shown, a silver spiral is formed.

Similar spirals can be constructed for all the metallic ratios by changing the value of n. These spirals are the approximations of the familiar logarithmic spiral.

The Metallic Polygons

This property is not yet generalized for all the metallic ratios. So far, only the golden and silver ratios have been observed in polygons.

Consider a regular pentagon of any size. The ratio between the sides of the triangle formed by any two adjacent sides and the diagonal is equal to ϕ, the golden ratio.

Something similar happens in a regular octagon. The ratio between the second diagonal of a regular octagon to its side is exactly equal to σ, the silver ratio.

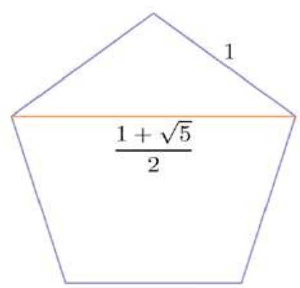

Figure 5.7: Golden Ratio in a regular Pentagon

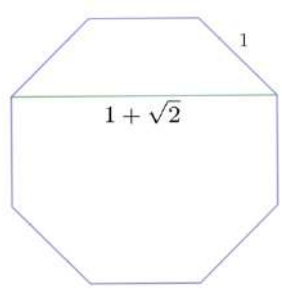

Figure 5.8: Silver Ratio in a regular Octagon

Applications

The Golden Ratio, as we know, exists in nature in many forms. It finds many applications in Technology and Engineering and is being used since ancient times. It has been used extensively in the construction of structures like the Egyptian Pyramids. Photography and fine arts also benefit from the Fibonacci sequence and the Golden Ratio. The paper sizes under ISO 216 are silver rectangles. Metallic ratios are extensively used to prepare beautiful paintings and art work. These ratios appear in nature at various places. Nature often uses spiral patterns for the efficient storage of the particles.

So, the next time someone talks to you about the Golden Ratio, do not forget to introduce them with the elegance of the entire family of metallic ratios.

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ON THE CALCULUS OF VARIATIONS

Daksh Dheer
Year II

ABSTRACT
In this paper, we introduce the field of variational calculus, which is concerned with finding the extrema of functionals: mappings from a set of functions to the real numbers. In other words, it answers the question: given a set of initial conditions, which *function* is the most optimal for the given problem? In this paper, we first derive the Euler-Lagrange equation. Then, we use it to prove a well-known fact about planes, and finally move on to solving Johann Bernoulli's Brachistochrone problem.

Keywords: *Brachistochrone, Euler-Lagrange Equation, Functional optimization, Geodesics, Variational Calculus*

INTRODUCTION

The calculus of variations is concerned with the discovery of extrema, and can thus be considered a subfield of optimization. However, the issues and solutions in this area are significantly different from those involving the extrema of functions of many variables, owing to the domain's nature on the quantity to be optimised.
A **functional** is a relationship between a collection of functions and the real numbers. The calculus of variations is concerned with the discovery of extrema for functionals rather than functions. Thus, the candidates for an extremum are functions rather than vectors in $\mathbb{R}^n$, lending the subject a different character. The functionals are typically defined by definite integrals; the sets of functions are frequently defined by boundary constraints and smoothness requirements introduced during the problem/model development.

THE EULER-LAGRANGE EQUATION

The central problem of the calculus of variations is to find extrema of functionals. The essential notion is as follows: replace the unknown function f in the functional $\mathcal{F}\{f\}$ by

$f + \eta$, where $\eta(x)$ is a "small" function. The variation of $\mathcal{F}$ under this change is

$$\delta\mathcal{F} = \mathcal{F}(\{f+\eta\}) - \mathcal{F}(\{f\});$$

If f is the function that defines the functional extremal, the variation in η must disappear to first order. Consider a functional that is entirely dependent on $f(x)$, $f'(x)$, and explicitly on x:

$$\mathcal{F}(\{f\}) = \int_a^b \Phi\left(f(x), f'(x), x\right) dx$$

Clearly,

$$\begin{aligned}\delta\mathcal{F}(\{f\}) &= \int_a^b \left[\Phi\left(f(x)+\eta(x), f'(x)+\eta'(x), x\right) - \Phi\left(f(x), f'(x), x\right)\right] dx \\ &= \int_a^b \left[\frac{\partial\Phi}{\partial f}\eta + \frac{\partial\Phi}{\partial f'}\eta' + \ldots\right] dx\end{aligned}$$

After that, we zero off the linear terms. We are now confronted with the challenge of assuming we know the "little" deviation η, but in reality we have no idea what its derivative is. The goal is to eliminate the derivative by re-expressing the term containing $(\eta)'$. For this, we employ integration by parts:

$$\begin{aligned}\int_a^b \frac{\partial\Phi}{\partial f'}\eta'(x)dx &= \int_a^b \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\eta\right) dx - \int_a^b \eta\frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right) dx \\ &= \left(\frac{\partial\Phi}{\partial f'}\eta\right)\Bigg|_a^b - \int_a^b \eta\frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right) dx\end{aligned}$$

We now see that in the majority of circumstances, we wish to specify the values $f(a)$ and $f(b)$. For instance, in the brachistochrone question that follows, the end points (x, y) are fixed. As a result, the function $\eta(x)$ must disappear at the interval's endpoints, and therefore the perfect derivative term vanishes:

$$\left(\frac{\partial\Phi}{\partial f'}\eta\right)\Bigg|_a^b = 0$$

and we can write, to first order in η,

$$\delta\mathcal{F} = \int_a^b \left[\frac{\partial\Phi}{\partial f} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right)\right]\eta(x)dx = 0$$

Now the variation $\eta(x)$ can be any function that vanishes at $x = a$ and $x = b$. Hence, it is quite arbitrary. This means that it must be the case such that

$$\frac{\partial\Phi}{\partial f} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right) = 0$$

This is called the **Euler-Lagrange equation**, and is vital to the calculus of variations, as we shall see in the successive sections.

GEODESIC ON A PLANE

As a first application, let us consider a well-known problem whose solution is widely known, to get a sense for how to use the Euler-Lagrange equation.
Let be a surface, and let p_0, p_1 be two distinct points on . The geodesic problem concerns finding the curve(s) on with endpoints p_0, p_1 for which the arclength is minimum. A curve having this property is called a **geodesic**.
Using calculus of variations, we shall show the well-known fact that the solution to the geodesic problem, on a plane, is a straight line, i.e., given two points on a plane, the shortest route connecting them is a straight line.
Let (x_0, y_0) and (x_1, y_1) be two arbitrary points. The arclength of a curve described by $y(x), x \in [x_0, x_1]$ is given by

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx.$$

The geodesic problem in the plane entails determining the function y such that the arclength is minimum. If y is an extremal for J then the Euler-Lagrange equation must be satisfied; hence,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) - 0 = 0$$

i.e.,

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{ constant}$$

The last equation is equivalent to the condition that $y' = c_1$, where c_1 is a constant. Consequently, an extremal for J must be of the form

$$y(x) = c_1 x + c_2,$$

where c_2 is another constant of integration. Thus, the only extremal y is given by $y(x) = c_1 x + c_2$, which describes the line segment from (x_0, y_0) to (x_1, y_1) in the plane (as expected). This proves the claim.

THE BRACHISTOCHRONE

The history of the calculus of variations began with a problem set by Johann Bernoulli (1696) as a challenge to the mathematical community, specifically to his brother Jacob. Johann's problem was to **identify the form of a wire along which a bead initially at rest glides under gravity from one end to the other in the shortest amount of time.** The wire's ends are given, and the bead's motion is considered to be frictionless.
Let us use Cartesian coordinates to model the problem with the positive $z-$axis taken to be in the direction of gravitational force. Let (a, z_0) be the initial position and (b, z_1) denote the final position of the bead, such that $a < b$ and $z_0 < z_1$. We must determine, among the

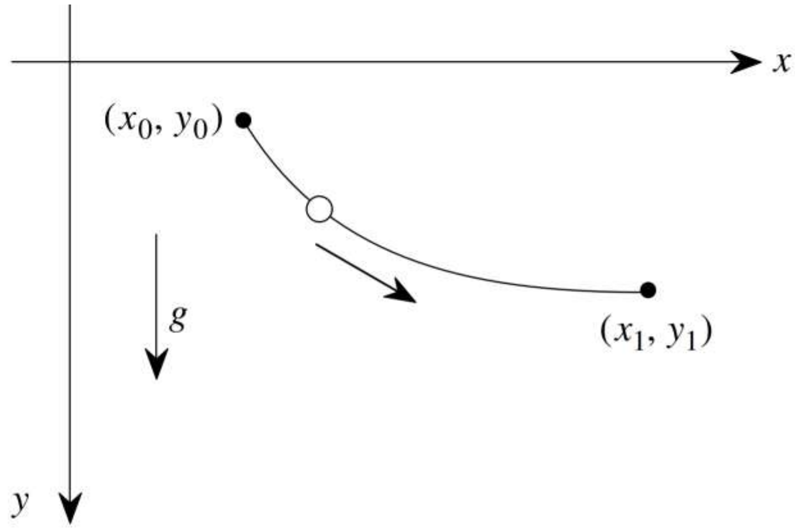

Figure 6.1: The brachistochrone problem

curves that have (a, z_0) and (b, z_1) as endpoints, the curve on which the bead slides down in the smallest amount of time. We know that the velocity of an object, starting at zero initial velocity, is given by

$$v(x) = \sqrt{2gz(x)}$$

for all x. Now we denote by $s(x)$ the length of the path from 0 to $(x, z(x))$. Then

$$s(x) = \int_0^x \sqrt{1 + z'(\hat{x})^2} d\hat{x},$$

implying that

$$\frac{ds}{dx} = \sqrt{1 + z'(x)^2}$$

Moreover, the length L of the whole path is given by

$$L = \int_a^b \sqrt{1 + z'(x)^2} dx$$

Now we switch from the space variable x to the time variable t. By definition of velocity, we have

$$v(t) = \frac{ds}{dt}$$

or

$$\frac{dt}{ds} = \frac{1}{v(s)}$$

Therefore, if we denote by T the total travel time, we obtain (after some changes of variables)

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)} ds = \int_a^b \frac{1}{\sqrt{2gz(x)}} \sqrt{1 + z'(x)^2} dx.$$

Thus we can formulate the brachistochrone problem as the minimization o functional

$$F(z) := \int_a^b \frac{\sqrt{1 + z'(x)^2}}{\sqrt{2gz(x)}} dx$$

subject to the constraints $z(a) = z_a$ and $z(b) = z_b$.
The function Φ is

$$\Phi\left(z, z'\right) = \sqrt{\frac{1 + \left(z'(x)\right)^2}{z(x)}}$$

(the factor $2g$ is immaterial) and the Euler-Lagrange equation becomes

$$\begin{aligned}
&\frac{\partial \Phi}{\partial z} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial z'}\right) \\
&\equiv -\frac{1}{2}\sqrt{\frac{1 + \left(z'(x)\right)^2}{z^3(x)}} - \frac{d}{dx}\left(\frac{z'(x)}{\sqrt{z(x)\left[1 + \left(z'(x)\right)^2\right]}}\right) = 0
\end{aligned}$$

To simplify the differential equation, we take z for the variable of integration. Then letting $\dot{x} = dx/dz$ we have

$$\delta \int_{z_a}^{z_b} dz \sqrt{\frac{1 + \dot{x}^2}{z}} = 0$$

or (since there is no explicit dependence on x)

$$\frac{d}{dz}\left(\frac{\dot{x}}{1 + \dot{x}^2}\sqrt{\frac{1 + \dot{x}^2}{z}}\right) = 0.$$

This can be integrated immediately, and squared, to give the first-order equation

$$\frac{1}{z} \cdot \frac{\dot{x}^2}{1 + \dot{x}^2} = A^2$$

which leads to the separable form

$$dx = \pm dz \sqrt{\frac{A^2 z}{1 - A^2 z}}$$

We want the $-$ sign because on physical grounds the altitude $z(x)$ must decrease monotonically with x. The solution is the equation of a cycloid,

$$A^2 x + B = \sqrt{A^2 z\left(1 - A^2 z\right)} - \sin^{-1}\sqrt{A^2 z}$$

where the constants A^2 and B must be adjusted to match the initial and final conditions. Thus, we have solved the Brachistochrone problem!

Conclusion

In this paper, we showed the use of variational calculus to derive the Euler-Lagrange equation, which has widespread utility over all areas of science. We also conclusively proved facts about geodesics and presented a solution to the brachistochrone problem. Thus, the calculus of variations is an important and elegant field of mathematics with a lot of applications.

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DECRYPTING THE MATHEMATICS OF CRYPTOGRAPHY

Simran Singh
Year II

Abstract

Have you ever thought of yourself as Bond 007 trying to solve secret codes? This article talks about the application of mathematics (particularly, number theory) in cryptography. *Cryptography* is the technological know-how of *secret writing*. It is the technique of securing and studying through the use of codes to protect the data from unwanted eyes. In this process, we start with the unencrypted data called *plain text*. It is then *encrypted* into *ciphertext* which in turn will be *decrypted* back to *plain text*. This science of using mathematics to encrypt and decrypt the data is based upon the type of cryptographic scheme being employed and some form of key and thus uses various aspects of mathematics including probability, statistics, abstract algebra, and so on.

Keywords: *Encryption, Decryption, Ciphers, Keys, Algorithms*

Introduction

The history of cryptography dates back to 1900 B.C. whose evidence has been found in inscriptions of old Egypt's tomb. **Arthashastra** by Kautilya conjointly offer bound clues and mention giving assignments to state spies in 'secret writing'. The most evident use of this technique was done by the well-known king Julius Caesar in wars, now popularly known as **Caesar Cipher** [7]. Developed around 100 BC, Caesar shifted each letter in his message, three letters to the right to produce what may well be referred to as 'ciphertext'. This gave him a strategic advantage, as his enemies could not understand the message. The increase in computer technology and the use of the internet has led us to focus a lot on our security and communication systems. This increasing reliance on online media and data storage increased the demand for advancement and learning in the science of cryptology.

Types of Cryptography

1. **Hash Functions:** No key is used, a hash value with a fixed length is calculated from the plain text. It is a *one-way encryption*, and thus cannot be decrypted to search out the particular string that created it. It is used to give a digital fingerprint to a file so that its contents cannot be tampered with. [5]

2. **Symmetric Key:** There is a *single common key* for both encryption and decryption. This can be understood by the following:

   For example, let the key for cipher be *k* chosen from a set of possible keys *K* to encrypt the plain text *m* chosen from the space of possible messages *M* and the result is ciphertext *c* from all possible values *C*.[1]

   Then *encryption* can be written as a function

   $$e : K \times M \to C$$

   where domain $K \times M$ is the set of pair *(k,m)* and range is the space of all ciphertexts *C*. Similarly, *decryption* is

   $$d : K \times C \to M$$

   Mathematically, this could be expressed as:

   $$d(k, e(k, m)) = m, \text{ for all } k \in K, m \in M$$

   Thus we can say that, for each key *k*, we have functions

   $$e_k : M \to C \quad \text{and} \quad d_k : C \to M$$

   satisfying the decrypting property

   $$d_k(e_k(m)) = m, \text{ for all } m \in M$$

   Thus, given one or more pairs of plain text and their corresponding ciphertexts, $(m_1, c_2), (m_1, c_2) \cdots (m_n, c_n)$, it is difficult to decrypt a *c* that is not within the list while not knowing the key *k*.

3. **Asymmetric Key:** *Two keys* that are mathematically related are used, one for encryption and the other for decryption. This process depends on the existence of *one-way functions* (functions that are easy to compute in comparison to their inverses, that are rather tougher).
   One may think about it as compounding paint colours: mixing them is straightforward however to retrieve back the constituent colours could be a difficult process.
   The trick is to find the trap door, which makes the inverse calculation easy!
   Mathematically, this could be written as:
   Here, the element *k* of key space is actually a pair of keys

$$k = (k_{pub}, k_{priv})$$

   For each *public key* and *private key*, we have corresponding *encryption* and *decryption* functions

$$e_{k_{pub}} : M \to C \quad \text{and} \quad d_{k_{priv}} : C \to M$$

   and thus

$$d_{k_{priv}}(e_{k_{pub}}(m)) = m, \qquad \text{for all } m \in M$$

   Therefore despite the fact that Eve has access to k_{pub}, she cannot decrypt the message without the information of k_{priv}.

SOME FAMOUS CIPHER PROBLEMS

In the subsequent section, we will be looking at some algorithms and techniques that are normally utilised in cryptography. We will also state certain theorems that are pre-requisite for the answers to those problems. The algorithms are in increasing order of mathematical knowledge required. With each problem, we will look into the drawbacks of the cryptographic system that paved the way for a better, more sorted, and *"difficult to decrypt"* system.
Before we begin, allow us to not forget two people, *Bob* and *Alice*, who wish to share messages at the same time as *Eve* (symbolizing the public) wish to intervene in between. These are the general variables used frequently while describing any cryptographic system.

Substitution Ciphers

It is an example of *symmetric-key cryptography*. In this process, every letter in the message is replaced by another letter in accordance with a given translation. It's among the very initial algorithms that were used for secret writing.

Sample Problem: Suppose in an algorithm, the *Plaintext A* is coded as *Ciphertext C*. We code *B* as *E* (2 beyond C), *C* as *G* (2 beyond E), and so on. Now through this algorithm, Bob desires to send a message to Alice, say ***GO THEE***. How can this be expressed mathematically? [6]

Mathematical Interpretation: It is apparent that all letters are being shifted and then substituted. This can be expressed in the following form:

$$c = 2p + 1$$

where c = ciphertext, p = plaintext

The factor 2 increases each *c* by *2* for every, one increase in *p*. The constant 1 has been chosen as $p = 1(A)$ corresponding to $c = 3(C)$.

And as you realize that there are finite letters in English, the cycle will hold on repeating itself. This can be expressed when we make the use of remainders when divided by 26 i.e., *congruence mod 26* (26 letters in the English language).
Hence, the formula came out to be:

$$c \equiv 3p + 1 \pmod{26} \qquad 0 \leq c \leq 25$$

Now to code the message, we can form a table, similar to the one given below:

Plaintext	***p***	***c=2p+1***	***c mod 26***	***Ciphertext***
G	7	15	15	O
O	15	31	5	E
T	20	41	15	O
H	8	17	17	Q
E	5	11	11	K
E	5	11	11	K

Hence, the ciphertext is: **O E O Q K K**

Remark: In such substitution processes, $26 \cdot 25 \cdot 24 \cdots 3 \cdot 2 \cdot 1 = 26!$ alternatives are possible to decipher the message! Hence, it will take years, for any outsider to discover the specific key.

But despite this large number, it is actually easy to break them! Confused? Here's the key!
The rationale is, "the frequency and likelihood of letters occurring within the ciphertext". Based on observations, it has been seen that certain letters occur more frequently as compared to others (for example, E repeats itself a lot). We can make a frequency table and through *statistical analysis* and *frequency count*, it is easy for an expert to decrypt the message and thus this was the biggest drawback of this process.

Diffie-Hellman Key Exchange

Problem: Suppose that Alice and Bob wish to share a message, however their means of communication are insecure and all pieces of information are visible to Eve. So how is it feasible that without an exchange of key, Alice and Bob are able to share and decipher each other messages? It was the brilliant insight of Diffie and Hellman which made this algorithm, the staple for any kind of cryptography at all.
The arithmetic behind Diffie Hellman is typically *modulo arithmetic.* More than encryption, the principle motive of this algorithm is to search for a solution for sending keys whilst all your exchanges are being monitored by a third party. The method for it is as follows:

- Alice and Bob ought to agree on a large prime number, say p and a non-zero integer g. Both of these are known to Eve.

- Now both Alice and Bob choose a private integer respectively which they do not reveal to anyone, say Alice took a and Bob chooses b.
  They both use it to compute

$$A \equiv g^a \pmod{p} \quad \text{and} \quad B \equiv g^b \pmod{p}$$

- They subsequently share these computed values through the public channel.
- Now again, Bob and Alice use their private integers to compute:

$$A' \equiv B^a \pmod{p} \quad \text{and} \quad B' \equiv A^b \pmod{p}$$

- Both the values A' and B' are actually the same because

$$A' \equiv B^a \equiv (g^b)^a \equiv g^{ab} \equiv (g^a)^b \equiv A^b \equiv B' \pmod{p}$$

Remark: Eve knows g,p and also A,B. Hence, she knows the value of g^a and g^b. To calculate a and b from this, she needs to apply the *Discrete Logarithm Problem* which is the problem of finding an exponent x, such that

$$g_x \equiv h \pmod{p}$$

This number x is called the *"discrete logarithm of h"* to the base g.

Eve can calculate the secret value g^{ab} by using the DLP, however that is not always a precise problem and calls for a chain of operations and calculations to be resolved, particularly when the value of given numbers is extremely high. Hence, the security of shared keys depends on the difficulty of DLP.

RSA Public Key Cryptosystem

Some vital theorems and concepts of number theory and abstract algebra forms the base of one of the most common cryptosystems widely used all over the network. The foundation of this system is based upon the fact that given a composite number, it is difficult to know its constituent prime factors. The prerequisite mathematical knowledge required for this is given as follows: [2]

- ***Fermat's Little Theorem:*** If p is a prime number, then for any integer a, the quantity $ap - a$ is an integer multiple of p i.e.,

  $$a^p \equiv a \pmod{p}$$

  In a special case, when a is not divisible by p, it is equivalent to

  $$a^{p-1} \equiv 1 \pmod{p}$$

- ***Greatest Common Divisor:*** GCD is the largest integer that divides two numbers GCD of *co-prime numbers* is 1.

- ***Multiplicative Inverse:*** Let a be an integer, then $a{\cdot}b \equiv 1 \pmod{m}$, iff $gcd\,(a, m) = 1$ Also if,

  $$a \cdot b_1 \equiv a \cdot b_2 \equiv 1 \pmod{m}, \text{then } b_1 \equiv b_2 \pmod{m}$$

  and b is the *multiplicative inverse* of a.

- ***Euler's Totient:*** $\phi(n)$ is defined as the number of elements in the set of all numbers which are less than n and relatively prime to n.
  For *prime numbers*, it is defined as $\phi(p) = p - 1$. For a product of prime numbers, it can be thus written as $\phi(pq) = (p - 1)(q - 1)$.

- ***Proposition:***
  This proposition has been derived from lots of mathematical concepts, thus just stating its statement which forms the foundation of RSA. [1]

  Let p and q be two distinct prime numbers and $e \geq 1$, satisfy the property $gcd\,(e, (p - 1)(q - 1)) = 1$
  Through inverse property, we know that e has an inverse modulo $(p - 1)(q - 1)$, say

$$de \equiv e \pmod{(p-1)(q-1)},$$

Then the congruence $x^e \equiv c \pmod{(pq)}$, has a *unique solution*

$$x \equiv c^d \pmod{(pq)}$$

Now, let us move on to the process of key generation and problem-solving.

Key Generation

- Let Bob choose two private prime numbers, say $p = 1229$ and $q = 1783$.
  Bob computes $N = pq = 1229 \cdot 1783 = 2191307$ and make this public.

- He calculates $\phi(N) = \phi(pq) = (p-1)(q-1) = 1228 \cdot 1782 = 2188296$

- He additionally chooses an encryption exponent e such that $0 < e < \phi(N)$ and $gcd(1, \phi(N)) = 1$. Let e in this case be 948047.

Key Encryption

- Alice converts her text message to an integer m such that $1 \leq m < N$, say 17007.
- She uses Bob's k_{pub} i.e., $(N, e) = (2191307, 948047)$ to compute

$$c \equiv m^e \pmod{N} \implies c \equiv 17007^{948047} \equiv 468067 \pmod{2191307}$$

- Alice sends the ciphertext $c = 468067$ to Bob.

Key Decryption

- Bob has $\phi(N)$ and e, thus he calculates

$$ed \equiv 1 \pmod{\phi)(N} \implies 948047.d \equiv 1 \pmod{2188296} \implies d = 2177759$$

- Then he takes $c = 4608067$ and compute

$$c_d \pmod{N},\ 468067_{2177759} \equiv 17007 \pmod{2191307}$$

- Therefore, Bob decrypted the original message i.e., 17001.

Remarks: The calculations done above are on small numbers, but in reality p and q have hundreds of digits and thus it would take years for even a super computer to find p and q, when only N and e are made public.

Conclusion

This paper aims to understand the mathematical ideas that are being employed within the field of cryptography. The application of this field in our lives is growing along with the advancement in technology, and hence it is interesting to decode the mathematics behind it.

When you connect to any secure website, the padlock icon means that your system is using public-key cryptography to verify the server, key exchange, and symmetric encryption to protect communication. Whether you are buying something online, sending messages to friends, or just browsing, cryptography makes all of this safe and secure for you.

In practice, asymmetric ciphers tend to be considerably slower (due to massive bit array) than symmetric ciphers such as *DES* and *AES*.

Many algorithms are primarily based on the idea that *lock* and *unlock* are inverse processes. *Diffie Hellman* and *RSA*, the two most widely used systems are based on *one-way function concept*. Their brief idea has been discussed, omitting their inverse calculation and proofs that are comparatively difficult to process, serving the aim of one-way functions. Once you dive into this subject, there's a great deal to learn, this paper is simply to present you with a quick insight into this field.

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ON THE ELEGANT UTILITY OF GENERATING FUNCTIONS

Daksh Dheer
Year II

Abstract

A **generating function** is a tool for encoding an infinite sequence of numbers by treating them as the coefficients of a power series. It is a bridge between discrete mathematics and continuous analysis.

This paper attempts to introduce the concept of a generating function in a general way and explain how it used to solve certain problems algebraically, that is, the basis of its elegance and utility.

Keywords: *Generating functions, Counting change, Partitions of integers, Fibonacci sequence, Recursions*

Introduction

A **generating function** is a tool for encoding an infinite sequence of numbers by treating them as the coefficients of a power series. It is a bridge between discrete mathematics and continuous analysis.
Herbert S Wilf puts it as

> A generating function is a clothesline on which we hang up a sequence of numbers for display.

Now, that is a marvelous quote because it is exactly how one should think about generating functions.
An alternate perspective can be to think of a generating function as a bag where all the important information about a sequence is stored and can be retrieved or 'generated' later on.
The whole power of generating functions is based around two simple facts:

1. *The product of x^p and x^q is x^{p+q}.*

2. *Knowing the coefficients of a polynomial or a power series allows us to manipulate it and extrapolate information.*

Getting started with Recursions

Consider the following recursive sequence, given that $a_0 = 3$:

$$a_{n+1} = 3a_n - 4 \tag{8.1}$$

Writing the first few terms, we get:
$a_1 = 5, \quad a_2 = 11, \quad a_3 = 29, \quad a_4 = 83, \quad a_5 = 245$, and so on ...
How do we solve this recursion to gain an explicit form? Well, we can look at it and notice that there is a rough correspondence of the terms of the sequence with powers of 3. Moreover, looking at the first few terms, one can easily guess and check that the sequence is given by:

$$a_n = 3^n + 2, \ \forall n \in \mathbf{N} \cup \{\mathbf{0}\}$$

Now, let us solve the same question using a generating function. We shall define the function as:

$$A(x) = \sum_{n \geq 0} a_n x^n$$

Our strategy is to evaluate $A(x)$ and expand it as a series to be able to read out the coefficients.

To evaluate $A(x)$, we consider $A(x) - 3xA(x)$ and use the relation given in (8.1)

$$\begin{aligned} A(x) - 3xA(x) &= (a_0 + a_1x + a_2x^2 + \dots) - 3(a_0x + a_1x^2 + a_2x^3 + \dots) \\ &= a_0 + x(a_1 - 3a_0) + x^2(a_2 - 3a_1) + x^3(a_3 - 3a_2) + \dots \\ &= 3 + x(-4) + x^2(-4) + x^3(-4) + \dots \end{aligned}$$

$$\begin{aligned} A(x)(1-3x) &= 7 - \frac{4}{1-x} \\ A(x) &= \frac{7}{(1-3x)} - \frac{4}{(1-3x)(1-x)} \\ &= \frac{1}{1-3x} + \frac{2}{1-x} \\ &= (1 + 3x + 3^2x^2 + 3^3x^3 + \dots) + 2(1 + x + x^2 + x^3 + \dots) \\ &= 3 + (3^1+2)x^1 + (3^2+2)x^2 + (3^3+2)x^3 + (3^4+2)x^4 + \dots \end{aligned}$$

Hence, we obtain:

$$A(x) = \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} (3^n + 2)x^n \implies a_n = 3^n + 2, \ \forall n \in \mathbf{N} \cup \{\mathbf{0}\}$$

One might wonder why we took such a tedious route to solve the same question. After all, we generally aim to reach an efficient solution, right?
Well, it is true that the generating functions approach proved to be quite long here, but this was just an example to illustrate how it is used.
We cannot always guess and check a recursion to obtain a solution, which is what our next example will be.
We shall find an explicit, closed form for the n^{th} Fibonacci number.

THE FIBONACCI SEQUENCE

The Fibonacci Sequence is given by the recursion:

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1; F_0 = 0; F_1 = 1)$$

Consider the generating function:

$$\begin{aligned} G(x) &= \sum_{n\geq 0} F_n x^n = 0 + 1x + \sum_{n\geq 2} F_n x^n \\ &= x + \sum_{n\geq 2}(F_{n-1} + F_{n-2})x^n = x + \sum_{n\geq 2} F_{n-1}x^n + \sum_{n\geq 2} F_{n-2}x^n \end{aligned}$$

Re-indexing the infinite sums, we get the following:

$$\begin{aligned} G(x) = \sum_{n\geq 0} F_n x^n &= x + \sum_{n\geq 1} F_n x^{n+1} + \sum_{n\geq 0} F_n x^{n+2} \\ &= x + x\sum_{n\geq 0} F_n x^n + x^2 \sum_{n\geq 0} F_n x^n \\ &= x + xG(x) + x^2 G(x) \end{aligned}$$

Note: We changed the first sum to go from $n \geq 1$ to $n \geq 0$, because $F_0 = 0$.

Hence, we get:

$$G(x) - xG(x) - x^2 G(x) = x \implies G(x) = \frac{x}{1 - x - x^2} \tag{8.2}$$

We can just use the MacLaurin series expansion of (8.2) to obtain the coefficients.
The Mathematica code is: Series[x/(1 - x - x^2), {x, 0, 50}]

However, our goal is to find an explicit form, so we shall now apply Partial Fraction Decomposition on the right side of (8.2):

$$1 - x - x^2 = 0 \implies x = \frac{1 \pm \sqrt{5}}{2}$$

Now, for compactness, let us put $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$.

$$\begin{aligned} 1 - x - x^2 &= -(x^2 + x - 1) = -(x - \phi_1)(x - \phi_2) \\ &= -\phi_1\phi_2 \left(\frac{1}{\phi_1}x + 1\right)\left(\frac{1}{\phi_2}x + 1\right) \end{aligned}$$

Now, $\phi_1\phi_2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = -1$ and also, we have $\frac{1}{\phi_1} = -\phi_2$ and $\frac{1}{\phi_2} = -\phi_1$ Hence, we get the following:

$$1 - x - x^2 = -(-1)(-\phi_2 x + 1)(-\phi_1 x + 1) = (1 - \phi_1 x)(1 - \phi_2 x)$$

From here, it is easy to check that

$$\frac{x}{1 - x - x^2} = \frac{\frac{1}{\sqrt{5}}}{1 - \phi_1 x} - \frac{\frac{1}{\sqrt{5}}}{1 - \phi_2 x}$$

Hence, finally, we obtain the following:

$$G(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - x(\frac{1+\sqrt{5}}{2})} - \frac{1}{1 - x(\frac{1-\sqrt{5}}{2})}\right) \tag{8.3}$$

Then, letting $c = \frac{1}{\sqrt{5}}$, equation (8.3) transforms into the following:

$$\begin{aligned} G(x) &= c\left(\frac{1}{1 - \phi_1 x} - \frac{1}{1 - \phi_2 x}\right) \\ &= (c + c\phi_1 x + c\phi_1^2 x^2 + c\phi_1^3 x^3 + \dots) - (c + c\phi_2 x + c\phi_2^2 x^2 + c\phi_2^3 x^3 + \dots) \end{aligned}$$

Thus, we have:

$$G(x) = \sum_{n \geq 0} F_n x^n = \sum_{n \geq 0} c(\phi_1^n - \phi_2^n)x^n \implies F_n = c(\phi_1^n - \phi_2^n)$$

Hence, the expression for the n^{th} Fibonacci number is:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \ \forall\, n \in \mathbf{N} \cup \{\mathbf{0}\}$$

Counting Money Change

Let us use generating functions now to tackle a more *practical* problem in real life.

The question: **What are the number of ways in which one can make change for 2000 rupees?**

Well, to get started, let us think about the number of ways we can make change for 5 rupees. Obviously, we can only use 1 rupee, 2 rupee or 5 rupee coins.

To do so, we can take the following routes:

1. Five 1 rupee coins 1 1 1 1 1
2. Three 1 rupee coins and One 2 rupee coin 1 1 1 2
3. Two 2 rupee coins and One 1 rupee coin 2 2 1
4. One 5 rupee coin 5

Hence, there are **4 ways** to do this.

Now, we can see that coins put together side by side all add up to 5, but how do we *mathematize* this?

If we consider the coins as mathematical entities, it looks like they're being multiplied, meanwhile, in reality the value they hold comes from their sum.

Is there a way we can link a mathematical product to a sum? Well, here comes that old proposition we talked about earlier, about the power of generating functions: exponentiation!

We can consider the coin values as exponents and then 'multiply' the coins and obtain the sum via exponents.

Using our new definition, let us restate what we did above. We have to obtain the amount of 5 rupees, so we need to use coins of smaller denominations. This also implies we cannot go above the amount, for example, we cannot ever use three 2 rupee coins. Hence, we'll need, at most, two 2 rupee coins.
Hence, we can use the following number of coins:

1 rupee coins: 0, 1, 2, 3, 4, 5 $\implies$ Consider $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5$
2 rupee coins: 0, 1, 2 $\implies$ Consider $g(x) = 1 + x^2 + x^4$
5 rupee coins: 0, 1 $\implies$ Consider $f(x) = 1 + x^5$

These polynomials represent the possible number of coins.

To obtain our answer, we multiply them out and consider the coefficient of x^5:

$$(pgf)(x) = x^{14}+x^{13}+2x^{12}+2x^{11}+3x^{10}+4x^9+3x^8+4x^7+3x^6+\mathbf{4}x^5+3x^4+2x^3+2x^2+x+1$$

Now that we have gathered a foot hold, let us move on to the original question: evaluating the number of such combinations for 2000 rupees.

Proceeding as above, we need to find the coefficient of x^{2000} in the polynomial below:

$$P(x) = \left(\sum_{i=0}^{2000} x^i\right)\left(\sum_{i=0}^{1000} x^{2i}\right)\left(\sum_{i=0}^{400} x^{5i}\right)\left(\sum_{i=0}^{200} x^{10i}\right)\left(\sum_{i=0}^{100} x^{20i}\right)\left(\sum_{i=0}^{40} x^{50i}\right)\left(\sum_{i=0}^{20} x^{100i}\right)\left(\sum_{i=0}^{4} x^{500i}\right)\left(\sum_{i=0}^{1} x^{2000i}\right)$$

The required coefficient, computed using a Computer Algebra System (CAS), is 14170471581.

Hence, there are exactly **14170471581 ways** to make change for 2000 rupees.

The formula we used above only works for rupees 2000 and below; if we compute the whole 18000 degree polynomial, and see the coefficients for the terms after 2000, they will be inaccurate. To understand why this is true, consider the number of ways to make change for 2020 rupees.
Proceeding similarly, we know that our first polynomial would contain factors from x^1 all the way up to x^{2020}, but the first polynomial we used in the 2000 rupee case only had terms from x^1 to x^{2000}. Similar reasoning follows for the other polynomials, and hence, our $P(x)$ will undercount all the values greater than 2020.
We, thus, derived the infinite polynomial later to overcome this issue.

Thus, to obtain a formula that works for any n, we need to consider the following polynomial:

$$G(x) = \left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)\left(\sum_{i=0}^{\infty} x^{10i}\right)\left(\sum_{i=0}^{\infty} x^{20i}\right)\left(\sum_{i=0}^{\infty} x^{50i}\right)\left(\sum_{i=0}^{\infty} x^{100i}\right)\left(\sum_{i=0}^{\infty} x^{500i}\right)\left(\sum_{i=0}^{\infty} x^{2000i}\right)$$

This polynomial is a product of 9 infinite geometric series!
Hence, we can write:

$$G(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{20}}\right) \left(\frac{1}{1-x^{50}}\right)\left(\frac{1}{1-x^{100}}\right)\left(\frac{1}{1-x^{500}}\right)\left(\frac{1}{1-x^{2000}}\right)$$

One can expand this polynomial out through its Maclaurin series to obtain the coefficients up to any terms!

The 2001 at the end can be replaced by any number to obtain the required coefficient.
Implicitly, the n^{th} coefficient is given by: $\dfrac{d^n}{dx^n}G(1)$.

Partitions of Positive Integers

After solving that, let us take one final problem from the branch of Partition in Number Theory: **What is the number of ways in which you can partition a natural number into positive integral parts?**
For example, the number 5 can be partitioned in seven ways, viz:

$$(5),\quad (4+1),\quad (3+2),\quad (2+2+1),\quad (3+1+1),\quad (2+1+1+1),\quad (1+1+1+1+1)$$

There is a problem here: there exists no explicit formula to calculate such number of ways for arbitrary n. But, we can make a generating function and move forward from there.
A partition is determined uniquely by the number of 1s, 2s and so on, and thus we write one factor for each integer and consider coefficients in the product:

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)\dots(1+x^k+x^{2k}+x^{3k}\dots)$$

$$= \prod_{k=i}^{\infty}\sum_{i=0}^{\infty} x^{ik} = \prod_{k=1}^{\infty}\frac{1}{1-x^k}$$

We can evaluate this series' coefficients to obtain the number of partitions for any n.
The Mathematica code is:
Series[Product[1/(1 - x^i), {i, 10}], {x, 0, 50}]

Conclusion

In this paper, we have showed the mathematical elegance as well as the practical utility of the concept of generating functions and used it to provide solutions to four different problems, thus illustrating the importance of generating functions.

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CALCULATING π UPTO HIGHER PRECISION

Prachi Sinha
Year I

ABSTRACT
This paper describes one of the most efficient known methods of calculating π upto higher decimal places. The area of a circle can be calculated using the known formula and by integration method. The integration method uses the equation of the circle, which can be expanded using the Generalised Binomial Theorem. This paper describes how π can be calculated upto higher precision by equating the area found by these two methods.

Keywords: *Generalised Binomial Theorem, Falling Factorial, Riemann Sum*

INTRODUCTION

Throughout history several attempts have been made to calculate the exact value of Pi. The earliest known method is the Archimedes Method in which an n sided polygon is inscribed and circumscribed in a circle and the ratio of perimeters of the respective polygons with circumference of the circle gave the lower and upper bound for the value of Pi, hence the value of Pi was estimated. The whole process was very tedious and time taking until Sir Isaac Newton came into the picture and proposed a new method of calculation of Pi using calculus and the Generalised Binomial Theorem by expressing Pi as an infinite sum sequence.

GENERALISED BINOMIAL THEOREM

Binomial Theorem

The Binomial Theorem states:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

where $n \geq 0$ and $\left(\begin{array}{c} n \\ k \end{array}\right)$ is a positive integer known as Binomial coefficient.
On putting $y = 1$, the theorem becomes:

$$(x+1)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) x^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} + \cdots + \left(\begin{array}{c} n \\ n-1 \end{array}\right) x^1 + \left(\begin{array}{c} n \\ n \end{array}\right) x^0$$

$$= 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots + nx^0$$

In Newton's Generalised Binomial Theorem, the exponent $n \in \mathbb{R}^+$ (it applies for complex exponents as well but that is not relevant here). On taking a positive real integer as an exponent, the expansion becomes an infinite sum series because of the Binomial Coefficient in the Generalised Binomial Theorem.

Generalised Binomial Theorem

Statement:

$$(a+b)^r = \sum_{k=0}^{\infty} \left(\begin{array}{c} r \\ k \end{array}\right) a^{r-k} b^k, \quad r \in \mathbb{R}^+$$

Here the term $\left(\begin{array}{c} r \\ k \end{array}\right)$ is the Binomial Coefficient of the $k + 1^{th}$ term of expansion.

$$\left(\begin{array}{c} r \\ k \end{array}\right) = \frac{r(r-1)(r-2)\cdots\cdot[r-(k+1)]}{k!}$$

$$= \frac{(r)_k}{k!}$$

where $(r)_k$ is called falling factorial.

EQUATION OF A UNIT CIRCLE

A unit circle is a circle with radius of 1 unit. The general equation of a circle is given as:

$$(x-h)^2 + (y-k)^2 = r^2$$

where
x : x - coordinate of the point on the circumference of circle.
y : y - coordinate of the point on the circumference of circle.
h : x - coordinate of the centre of circle.
k : y - coordinate of the centre of circle.
r : radius of the circle.

For a unit circle, centred at the origin, the equation of circle becomes:

$$(x-0)^2 + (y-0)^2 = (1)^2$$

$$x^2 + y^2 = 1$$

AREA OF DIFFERENT REGIONS OF CIRCLE

Area of a circle is given by the product of Pi with its radius squared.

$$\text{Area of a circle } = \pi * r^2$$

Area using known Formulae

For a unit circle,

$$\begin{aligned}\text{Area } &= \pi * (1)^2 \text{ sq. units}\\ &= \pi \text{ sq. units}\end{aligned}$$

For the quadrant of circle,

$$\text{Area } = \frac{\pi}{4} \text{ sq. units} \qquad \ldots\ldots(1)$$

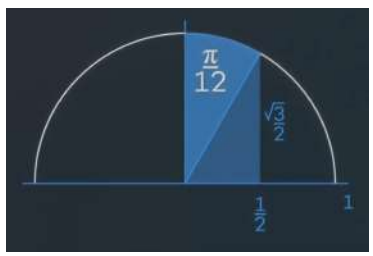

Figure 9.1: Area from $x = 0$ to $x = \frac{1}{2}$

Area of the shaded region

$$A = \text{ Area of Sector + Area of the Triangle}$$

$$A = \frac{\pi}{12} + \frac{1}{2} * \frac{\sqrt{3}}{4} \text{sq. units}$$

Hence, Area of the shaded region:

$$A = \frac{\pi}{12} + \frac{\sqrt{3}}{8} \text{sq. units} \qquad \ldots\ldots(2)$$

Area Of circle using Integration:

Consider the quarter of circle lying in the first quadrant.
Equation for this quarter is the equation of given circle:

$$\begin{aligned}&x^2 + y^2 = 1^2\\ &y^2 = 1 - x^2\\ &y = \pm(1 - x^2)^{\frac{1}{2}}\end{aligned}$$

The $(+)$ sign in the given equation represents the region of circle above the $x-$axis while the $(-)$ sign represents the region of the circle below $x-$axis. As we need only the quarter of circle in first quadrant, we can take the equation of circle
as:

$$y = \sqrt{1 - x^2}$$

Partitioning the given quadrant in rectangles of infinitesimally small area dA and width dx.
Length of each rectangle = $\sqrt{1 - x^2}$.
Taking the Riemann sum we get:

$$\text{Area} = \int dA$$

Area of each rectangle:

$$dA = \sqrt{(1 - x^2)}dx \qquad \ldots\ldots(3)$$

On Integrating both sides and putting limits ,we have area of the quadrant as:

$$\int_0^1 dA = \int_0^1 \sqrt{1 - x^2}dx \qquad \ldots\ldots(4)$$

Similarly, the area of shaded region in Fig 9.1 can be calculated by integrating on both sides of (3) and taking limit from $x = 0$ to $x = \frac{1}{2}$.

$$\int_0^{\frac{1}{2}} dA = \int_0^{\frac{1}{2}} \sqrt{1 - x^2}dx \qquad \ldots\ldots(5)$$

CALCULATING π

In the previous section, we find area of circle using two different methods.
On Equating RHS of (1) and (2), we get:

$$\frac{\pi}{4} = \int_0^1 \sqrt{(1 - x^2)}$$

which can be rewritten as:

$$\frac{\pi}{4} = \int_0^1 (1 - x^2)^{\frac{1}{2}} \qquad \ldots\ldots(6)$$

On Expanding the RHS of equation (6) using the generalised binomial expansion, we get:

$$\frac{\pi}{4} = \int_0^1 \left[1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots\right] dx$$

$$\pi = 4\left[x - \frac{1}{2} * \frac{x^3}{3} - \frac{1}{8} * \frac{x^4}{4} - \frac{1}{16} * \frac{x^7}{7} - \frac{5}{128} * \frac{x^9}{9} - \cdots\cdots\right]_0^1$$

Now, we have π as an infinite sum series which requires only basic arithmetic operations to calculate its value to an arbitrarily high precision.
Note: We are able to integrate the above power series term-by-term because it is uniformly convergent.
On further evaluation, we get:

$$\pi = 4\left[1 - \frac{1}{2} * \frac{(1)^3}{3} - \frac{1}{8} * \frac{(1)^4}{4} - \frac{1}{16} * \frac{(1)^7}{7} - \frac{5}{128} * \frac{(1)^9}{9} - \cdots\right]$$

$$\pi = 4\left[1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \cdots\right]$$

On calculating just first five terms of the series, we get the value of π correct upto 2 decimal place.

$$\pi = 3.145$$

On equating RHS of (2) and (5), we get

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \int_0^{\frac{1}{2}} \sqrt{1 - x^2}$$

Rewriting the equation as:

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \int_0^{\frac{1}{2}} (1 - x^2)^{\frac{1}{2}}$$

Expanding RHS using Generalised Binomial Theorem

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \int_0^{1/2} \left(1 - \frac{1}{2}\pi^2 - \frac{1}{8}\pi^4 - \frac{1}{16}\pi^6 - \frac{5}{128}\pi^8 - \cdots\right) dx$$

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \left[\frac{1}{2} - \frac{1}{2} * \frac{1}{3} * \left(\frac{1}{2}\right)^3 - \frac{1}{8} * \frac{1}{5} * \left(\frac{1}{2}\right)^5 - \cdots\right]$$

On further evaluation, it can be written as:

$$\frac{\pi}{12} = \left[\frac{1}{2} - \frac{1}{2} * \frac{1}{3} * \left(\frac{1}{2}\right)^3 - \frac{1}{8} * \frac{1}{5} * \left(\frac{1}{2}\right)^5 - \frac{1}{16} * \frac{1}{7} * \left(\frac{1}{2}\right)^7 - \cdots\right] - \frac{\sqrt{3}}{8}$$

Taking first five terms of the expansion:

$$\pi = 12\left[\frac{1}{2} - \frac{1}{2} * \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{8} * \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{16} * \frac{1}{7}\left(\frac{1}{2}\right)^7 - \frac{5}{128} * \frac{1}{9}\left(\frac{1}{2}\right)^9 - \frac{\sqrt{5}}{8}\right]$$

$\Rightarrow$ The term $\sqrt{3}$ can be further expanded using the theorem as:

$$\sqrt{3} = \sqrt{4-1}$$

$$\sqrt{3} = 2\sqrt{1-\frac{1}{4}}$$

$$\sqrt{3} = 2\left(1-\frac{1}{4}\right)^{\frac{1}{2}}$$

$$\sqrt{3} = 2\left[1+\frac{1}{2}\left(\frac{-1}{4}\right)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-1}{4}\right)}{2!}+\cdots\right]$$

Using this, we get,

$$\pi = 3.14159$$

which is correct upto 5 decimal places.

CONCLUSION

By just evaluating first 50 terms of expansion, value of π can be found upto 25 decimal places. Hence, the time required to calculate Pi to higher precision, which goes way beyond any practical purposes and becomes more a matter of flexing muscles, is reduced exponentially using this method.

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CONWAY'S GAME OF LIFE

Sayon Das
Year I

ABSTRACT

This research paper is an homage to Dr. Conway, a brilliant Mathematician with a youthful mind who worked on the number theory while playing with numbers. He came up with numerous games and mathematical puzzles some of which still remain unsolved. Sadly, he could not survive the waves of COVID-19 and had his last breath oh April 8, 2020.

INTRODUCTION

John Horton Conway was an English man who aimed to be a mathematician when he was 11, so he did. In fact, he became "the world's most charismatic mathematician", a nickname he was popularly known by. Conway, as a mathematician, was deeply inspired by the works of John von Neumann. He considered Neumann's work on the concepts of cellular automation and the game theory very thorough and intriguing and hence he credits Neumann as the actual inspiration behind his invention - 'The Game of Life'. Conway, in the book called *Autonoma Studies* – a compilation of the works done on automation by Neumann, Claude Shannon, and Edward F Moose amongst many others, came across Neumann's self-replicating machine and his previous work on different cellular automaton rules. And thus began his quest to find a certain cell automaton that could do arbitrary computations and build copies of itself. He began doing experiments in 1968 and finally ended up creating a game that explored the idea of indefinite imitation on the basis of just 3 simple rules. Conway called this game 'Life', because it functions the same way life does. [1]

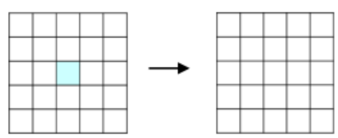
The Game of Life was first published by Martin Gardner, a famous writer and one of Conway's closest friends, in the October issue of the Scientific American, 1970. [2]

> This month we consider Conway's latest brainchild, a fantastic solitaire pasttime called 'life'.
>
> -Mathematical Games, Scientific American

The paper described the game to be an elegant mathematical model of computation — a cellular automaton, a little machine of sorts, with groups of cells that evolve from iteration to iteration, as a clock advances from one second to the next. The game unfolds on an infinite two-dimensional grid composed of cells, each either 'on' or 'off'. Well in simple terms, each cell is either 'alive' or 'dead' at a particular moment in time and based on a few mathematical rules, can live, die or multiply. It takes place in discrete time, with the state of each cell at time t determined by its state and the states of its eight immediate neighbours at the time, t-1. The game starts by placing a number of filled cells on a two-dimensional grid. Each turn then switches cells 'on' or 'off' depending on the cells' state surrounding it. All eight of the cells surrounding the currently alive cell are checked to see if they are 'on/alive' or not. Any cell that is 'on' is counted, and this count is then used to determine what will happen to the current cell. [3] [4]

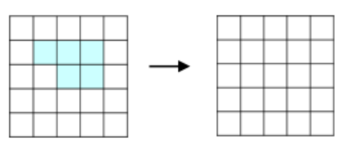
The Rules

1.**The Rule of Death:**
a. A cell with one or no neighbours vanishes, as if it just died of solitude.

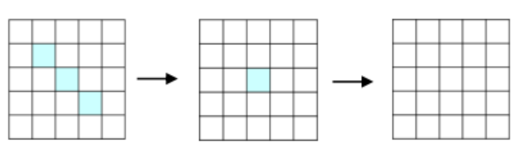


The highlighted cell in the first grid represents a live cell. However, since it has no neighbours, it vanishes in the next turn.

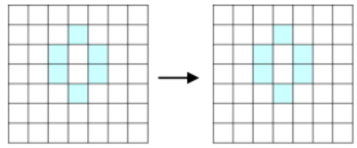
b. A cell with more than three neighbouring cells also dies, as if by overpopulation



2. **The Rule of Survival:** Only those cells that have exactly two or three cells around themselves survive.

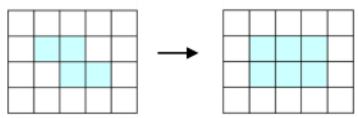


Only the centre cell has two neighbouring cells and hence it survives the first turn whereas the other two die. However, in the next turn, it too vanishes because of isolation.

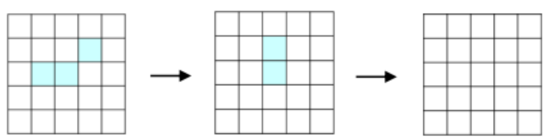


All the cells in this grid have exactly 2 neighbours and hence all survive and since the pattern remains the same without any change in the count of neighbours of every single cell, all the cells remain stable and live.

3. **The Rule of Birth:** If a cell with no life in it is in contact with three live cells, that cell will come to life.



The cells in the top right and bottom left corners are born as they are in contact with three live cells each.



In this case, the cells in the left and right disappear in the first turn as they are in contact with only one cell each. The one in the centre survives as it is in contact with two live cells and the one above it as is in contact with three live cells is born. [6]

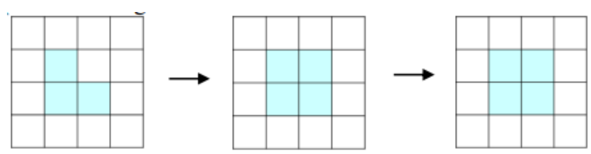
These rules imbibe the concept of life, death, and survival and hence the names. The rule of birth signifies that there has to be enough energy available to breed or multiply. The rule of death is reflective of the pain one goes through when they have no one to turn to. The loneliness eats them up and they cannot survive any longer and there comes the rule of survival. It signifies how to live life we need someone or the other to be there by us no matter how independent we are as an individual!

Conway always mentioned the game to be a 'no-player' game. That's because the game solely works on the basis of these pre-established rules. Sure, one has to initially place the first cell or as many first cells as they want, in any position they wish to, within the grid but once the game begins its just a work of computation of the rules to determine the fate of the cells placed and the ones that are yet to be born. The game of life was originally played (i.e., successive generations were produced) by hand with counters, but implementation on a computer greatly increased the ease of exploring patterns.

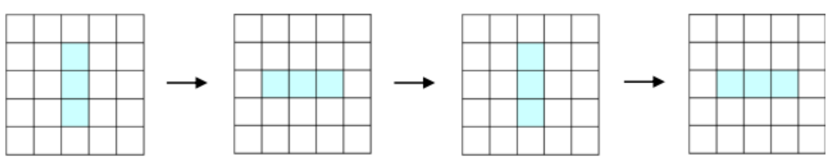
Patterns

Since its inception, there has been considerable interest in discovering novel patterns within the Game of Life. Patterns can be categorized according to the complexity of their behaviour, from simple unchanging 'still lives' to emulations of universal Turing machines (see below). Here are some examples of some Game of Life patterns at the simple end of the scale:

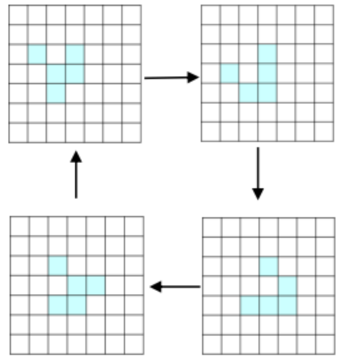
1. **Still Life:** These patterns once achieved don't change on their own. They can either be the initial configuration set by the player/ programmer and can also be achieved by some other initial configurations but once achieved they remain the same unless they are disturbed by other patterns. One of the simplest still-life patterns of the game is the block.



2. **Oscillators:** These are patterns that change but repeat themselves after a particular number of iterations (period). The example below shows *the blinker*, which is a period-2 oscillator.



3. **The Glider:** *The glider* as the name suggests glides through the grid wherein it repeats the pattern after every 4 iterations.



This glider holds a very strong position in history as it led to the invention of guns in cellular automation. Bill Gosper, in winning a bet from Prof. Conway, discovered the first glider gun in 1970. The existence of guns in the game meant that initial patterns with finite numbers of cells can eventually lead to configurations with limitless numbers of cells, something that John Conway himself originally conjectured to be impossible. [7]

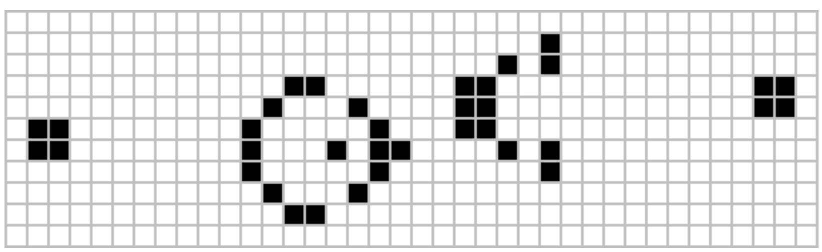


Fig 1: Gosper's Glider Gun [7]

The Glider Gun was very unique in the sense that it produced its first glider in the 15th generation, and another glider every 30th generation from then on. This discovery eventually led to the proof that Conway's Game of Life could function as a **'Universal Turing Machine'** i.e., it was 'Turing complete' (Berkelamp, Conway, Guy, 1982). Turing complete means that provided there is no absence in any constraints of memory or time, the Game of Life has unlimited computational power. More recently, Universal Turing Machines have been implemented in practice in Game of Life environments like the Rendell, 2000 (Fig 2). Another recent development, the 'Gemini' pattern, reaches back to Neumann's early interest in self-replicating automata.

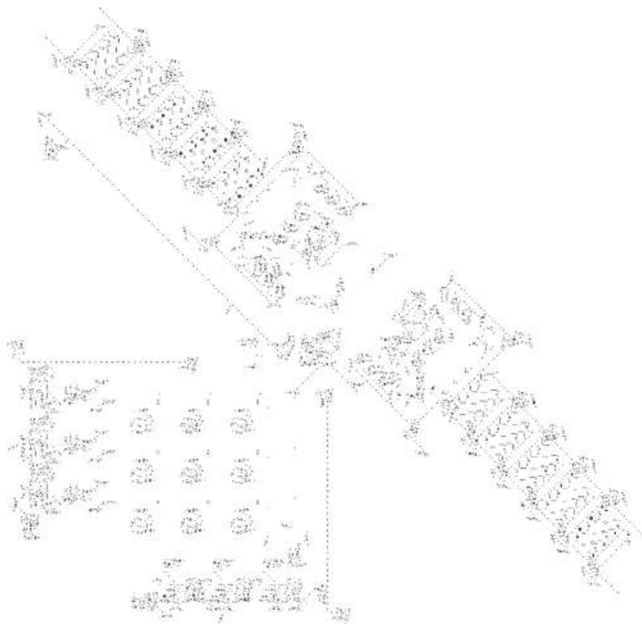


Fig 2: Rendell, 2000, A Universal Turning Machine implemented in Conway's Game of Life [5]

Patterns within the Game of Life can be much more complex than illustrated in the above examples, and can also be organized in ways that perform functional operations. Streams of gliders (and other moving objects, generally known as 'spaceships') can be considered as signals which have causal effects on other patterns with which they interact. These interactions can be organized to furnish basic computational procedures such as logic gates (AND, OR, NOT) as well as simple memory counters which complement the game being a Universal Turning Machine. [5]

Conclusion

The parallels of the game with real-life doesn't just end with the concepts of birth and death. But the one thing that connects the game to real-life the most is its unforeseeable nature. Life is unpredictable. Not everything is in our control but we eventually move on with life and accept those changes either with consent or by force before life finally ends. And that is exactly how this game works. No matter how many times we observe a particular pattern, compute and reckon the next possibilities ahead of their occurrence, we can never be sure of what pattern is to unfold next! Well, at times a few patterns do repeat themselves but that only happens until the moment they don't come in contact with another pattern and the possibility of that happening is only as much as of a person being able to blow the candles of his hundredth birthday! Patterns are destined to collide; obstacles are sure to come. If they don't, the cells sure did get lucky but no one can predict what awaits them next and that is what makes this game what it is - The Game of Life!

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GRANDI'S SERIES

Saloni Juneja
Year I

ABSTRACT
Guido Grandi, an Italian Mathematician, published his works in the book *Quadratura circula et Hyperbole per infinitas hyperbolas geometric exhibita*, which is briefly referred to as *Quadratura*, in 1703. This academic paper is an attempt to comprehend Grandi's Series' non-convergent nature and varied summability methods that were employed to assign the series a value.

Keywords: *Divergent series, Cesàro sum, Abel sum*

HISTORY

Grandi

In Mathematics, *Grandi's Series* is a term used for the infinite series

$$1 - 1 + 1 - 1 + \cdots = \sum_{n=0}^{\infty} (-1)^n.$$

It is a divergent series.
Note: We use the term *divergent* to denote any series which is not convergent.

Grandi discovered that putting parenthesis into $1 - 1 + 1 - 1 + \cdots$ yielded varying results:
either

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0$$

or,

$$1 + (-1 + 1) + (-1 + 1) + \cdots = 1$$

However, he didn't suppose it summed to either 0 or 1. Rather, he thought that the true value of the series was $\frac{1}{2}$. [2]

Further, Grandi also used the binomial expression

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

and substituted $x = 1$ to get

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

Leibniz

Grandi sent a replica of 1703 *Quadratura* to Gottfried Wilhelm Leibniz, who referred to it as a bromidic and less advanced attempt at calculus.
However, he believed that the argument from $\frac{1}{1+x}$ was valid, since the relation

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

holds for all $x < 1$, it ought to hold for $x = 1$.
Nonetheless, he thought that one should be able to find sum of the series $1 - 1 + 1 - 1 + \cdots$ directly while not having to check with expression $\frac{1}{1+x}$.

Taking an even number of terms, he discovered that the last term is -1 and the sum is 0. Using an odd number of terms, the final term is 1 and the total is 1.
Now, because the infinite series has neither an even nor an odd number of terms, it does neither yield 0 nor 1 as its "value", therefore it becomes something between 0 and 1. So he took the arithmetic mean of 0 and 1 i.e. $\frac{0+1}{2} = \frac{1}{2}$. [1]

Modern Mathematics

Ernest Cesàro, in 1890, proposed a concept, which sparked research into various summability methods for divergent series.

INTRODUCTION

Divergent Series

Divergent Series, for our intents and purposes, may be defined as an endless Sequence of Partial sums (SOPs) of a series that has no finite limit.

Grandi's SOPs are (1, 0, 1, 0, 1), and it does not approach any number (though it does have accumulation points at 0 and 1).
As a result, Grandi's series is divergent.

Certain such series can be given "values" in specialised mathematics. These methods of summation or summability are partial functions from a set of series to a value.
Cesàro summation, for example, lends value to the Grandi series. [3]

GRANDI'S PARADOX

In the *Quadratura*, Grandi considered the expression

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \tag{11.1}$$

that converges for $x < 1$.
A method of making series expansion in (1) convergent for $x > 0$, using a simple manipulation followed by a variable adjustment,

$$\frac{1}{1+x} = \frac{1}{x(\frac{1}{x+1})} = \frac{\frac{1}{x}}{\frac{1}{x+1}} = \frac{\tilde{x}}{1+\tilde{x}} = \tilde{x} - \tilde{x}^2 + \tilde{x}^3 - \tilde{x}^4 + \cdots \tag{11.2}$$

such that $\tilde{x} = \frac{1}{x}$ and $\tilde{x} < 1$ for $x > 1$.

Therefore, unlike (1), rearranged form (2) is convergent for $x > 1$.

For $x = 1$
LHS: $\frac{1}{1+1} = \frac{1}{2}$

RHS: $1 - 1 + 1 - 1 + 1 - 1 + \cdots$

Grandi's paradox was established in two different ways, both yielding two completely different outcomes:
1. Even number of terms

$$(1-1) + (1-1) + (1-1) + \cdots = 0$$

2. Odd number of terms

$$1 - (1-1) - (1-1) - (1-1) - \cdots = 1$$

From this, we tend to believe that re-grouping of divergent series isn't justified and might provide illogical results.

Somehow, Grandi's initial argument that the sum is $\frac{1}{2}$ appears intuitively acceptable. There is a different approach to demonstrate this as well.
Let Grandi's series be represented with G.

$$G = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Now,

$$\begin{aligned} 1 - G &= 1 - (1 - 1 + 1 - 1 + \cdots) \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= G \\ 1 &= 2G \\ G &= \frac{1}{2} \end{aligned}$$

The algebraic procedure described above, on the other hand, is used to evaluate convergent series in order to derive a third value.
To resolve these paradoxes, new summation procedures should be devised that offer a precise and formally proved sum. [8]

Summation Methods

Some formal and legal adjustments that resulted in $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$ were utilised for stability and linearity:

1. Sum by sum addition or subtraction of two series
2. Scalar multiplication of each term
3. Shifting the series without altering the sum
4. Adding a new term to the series to increase the sum

Cesàro summation and Abel summation are two methods for assigning 'sum' to Grandi's series that takes into account the above operations. [4]

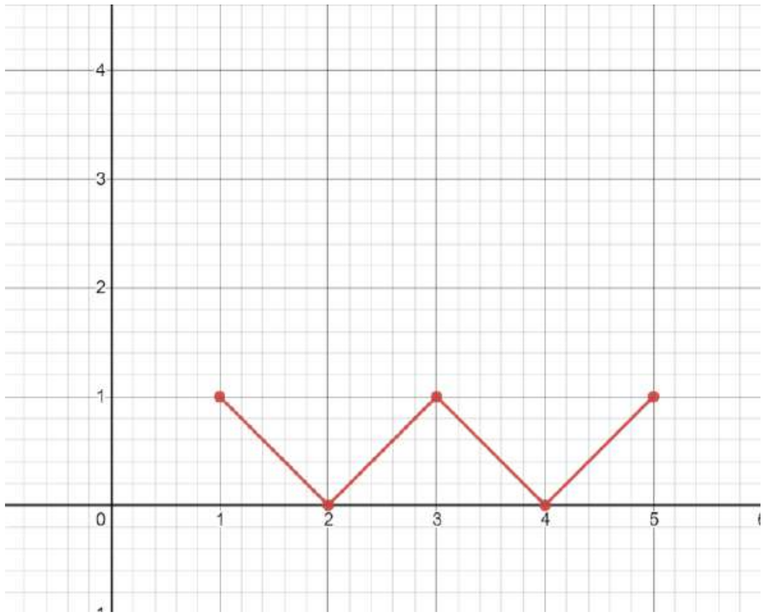

Figure 11.1: Partial Sum vs Number of Terms taken

Cesàro Sum

Ernest Cesàro, an Italian Mathematician, made this first rigorous attempt to access the Grandi series.
The Cesàro sum of a series is the limit as $n \to \infty$ of the sequence of arithmetic means of the first n partial sum of series.

Let G represent the Grandi series and $a_n = (-1)^n$ for $n \geq 0$

$$G = \sum_{n=0}^{\infty} a_n = 1 - 1 + 1 - 1 + \cdots$$

i.e., the sequence $(1, -1, 1, -1, 1, \ldots)$

Now, let $(s_k)_{k=0}^{\infty}$ denote partial sums of G and t_x denote the average of first x partial sums.

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + (-1) = 0 \\ s_3 &= 1 + (-1) + 1 = 1 \\ &\vdots \end{aligned}$$

The sequence of Partial sums $\to (1, 0, 1, 0, 1, \ldots)$

Further,

$$\begin{aligned} t_1 &= 1 \\ t_2 &= \frac{1+0}{2} = \frac{1}{2} \\ t_3 &= \frac{1+0+1}{2} = \frac{2}{3} \\ &\vdots \end{aligned}$$

The sequence of arithmetic mean of partial sums

$$\left(1, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \frac{4}{7}, \ldots\right)$$

or,

$$\left(\left(\frac{1}{2}+\frac{1}{2}\right), \frac{1}{2}, \left(\frac{1}{2}+\frac{1}{6}\right), \frac{1}{2}, \left(\frac{1}{2}+\frac{1}{10}\right), \frac{1}{2}, \left(\frac{1}{2}+\frac{1}{14}\right), \ldots\ldots\right)$$

where $t_x = \frac{1}{2}$ for even x

and, $t_x = \frac{1}{2} + \frac{1}{2x}$ for odd x

As a result, the sequence of arithmetic mean converges to $\frac{1}{2}$.

Hence, the Cesàro sum of the Grandi series is $\frac{1}{2}$.

Alternatively, the Cesàro limit of sequence $1, 0, 1, 0, 1, \ldots$ is $\frac{1}{2}$. [5]

ABEL SUM

Given a series,

$$a_0 + a_1 + a_2 + \cdots$$

one can produce a new series

$$a_0 + a_1 x + a_2 x^2 + \cdots .$$

If the later series converges for $0 < x < 1$ to a function with a limit as x approaches 1, this limit is referred to as the Abel sum of the first series, after Abel's Theorem, which ensures that the method is consistent with ordinary summing. [4]

For Grandi's series, one has

$$A\sum_{n=0}^{\infty}(-1)^n = \lim_{x\to 1}\sum_{n=0}^{\infty}(-x)^n = \lim_{x\to 1}\frac{1}{1+x} = \frac{1}{2}$$

Conclusion

Although the series is not convergent, it can be summed using a variety of approaches. Using all-natural computations, the most reasonable number to assign to the Grandi series is $\frac{1}{2}$. Even if it is not universally acknowledged, it should not be overlooked.

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EVOLUTION OF NUMBER SYSTEM

Jaya Narang
Year I

Abstract

This research paper aims to depict the importance of the encryption of data. We need to know how the presence of a number system enables easy conversion of numbers for technical purposes. The Egyptians were known to have invented their ciphered number system, followed by Greeks through mapping their counting numbers. In 500AD, Indian mathematicians introduced the symbol for zero. It was then that the number system became effective.

Keywords: *Hieratic numerals, Phonetic, Acrophony.*

Introduction

The number system or the numeral system, utilizing which we represent or name the digits, is a mathematical notation for presenting the arithmetic and algebraic structure of the figures. Numbers make an essential part of life. From knowing one's age to buying groceries using money, deciding best performance, and ranking an individual, we have to use numerals. This paper aims to define and understand the introduction of numbers and different types of number systems that are present and how we can use a number system to benefit and carry out discoveries.

History And Mathematics

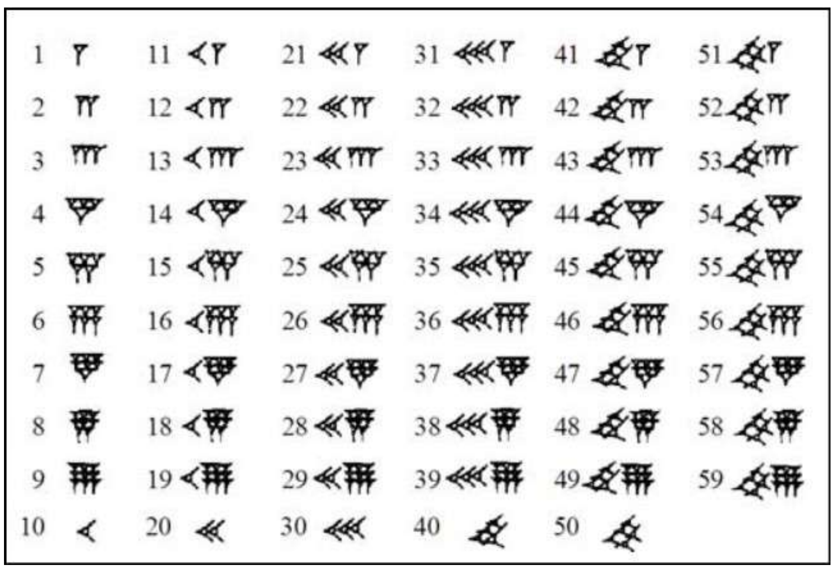
The history of mathematics revolves around explaining the evolution of number system. It relates to discoveries and the origin of mathematics and describes mathematical notations of the past. Further, we'll see how this enormous and most significant subject made a start and how it is used in the present to make new developments.

Journey Of The Evolution Of The Number Systems

The building blocks of a numeral system had started with the counting of fingers, which further evolved into sign language, moved further to hand-to-eye-to elbow communication of numbers and gave way to written numbers.

Tallies have supported counting, followed by small clay tokens known as first written numerals. We start by introducing the following number systems developed in the history of mathematics.

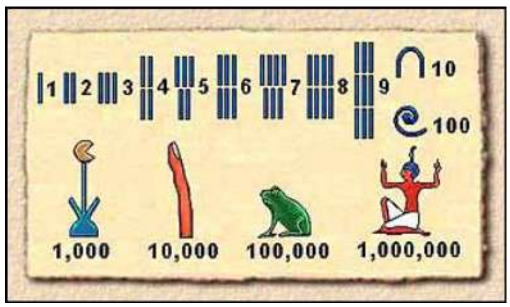
(i.) Babylonian Number System



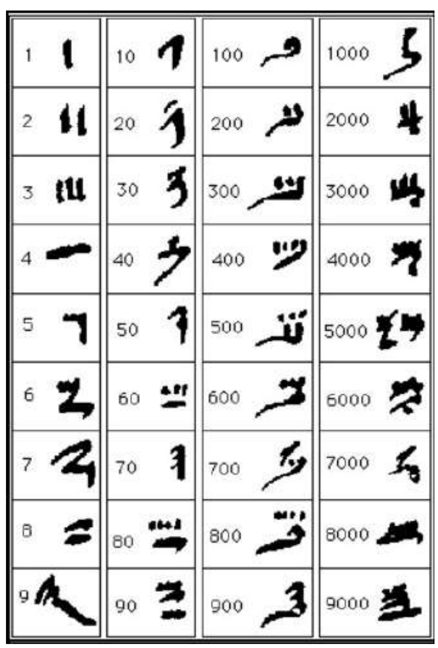
Ancient Sumerians were the ones who introduced and used this system to measure time, angles, and geographic coordinates. It is a very peculiar fact for all of us to learn that instead of writing a zero to depict empty spaces, they used doubled-angle wedges. The origin of this numeral system helped us know why there are 60 seconds in a minute, 60 minutes in an hour, and 360 degrees in a circle. Babylonians used 60 as a base value, which means it is a sexagesimal numeral system. This fact helps us evolve everything on this planet as a multiple of 60.

(II.) EGYPTIAN NUMBER SYSTEM: HIEROGLYPHIC NUMBERS AND HIERATIC NUMBERS

In Egyptian mathematics, the writing and counting system is "Hierology". This introduction was back in 3000BC. A Hierology is a pictorial representation of a word. This numeral system is the first to develop with the base as "10". A stroke represents a unit, a heel bone symbolizes tens, a coil of rope represents hundreds, and a lotus plant represents a thousand. Similarly, there are symbols for subsequent powers of 10. During the earlier times, when symbols were essential for counting huge numbers, it was a benefit as the architect, and all building structures could depict some value. So Egyptian numerals were an asset to the numeral value system.



The other numeral system discovered and used by Egyptians was Hieratic numerals. Writing on Papyrus, an Egyptian scroll that consisted of mathematical problems and tables, was the invention when the Hieratic numeral system came into the picture. In the system, numbers were represented in a compact form.

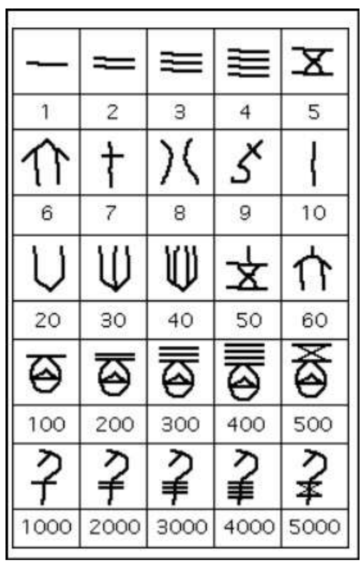


The evolution of the Egyptian numeral system was a boon. In the latter numeral system, the Hieratic numeral system, fewer symbols had represented a value.

(iii.) Chinese Number System

The arousal of the Chinese number system had a great story. In a village, Xiao Dun of the An-Yang district of Henan Province, there had been a discovery of "Bones" and "Tortoise Shells". They were inscribed with Chinese numerals. The cracks on one side of the Shell depicted some knowledge by ancestors.
These inscriptions had numerical information and told Chinese people about the number of men who lost in the battles, the number of days the battle took place, the number of sacrifices made, and all other relevant information of war in numerical form. Thus, the Chinese Numerical system came into existence.

This system was a decimal system and was multiplicative and additive also.



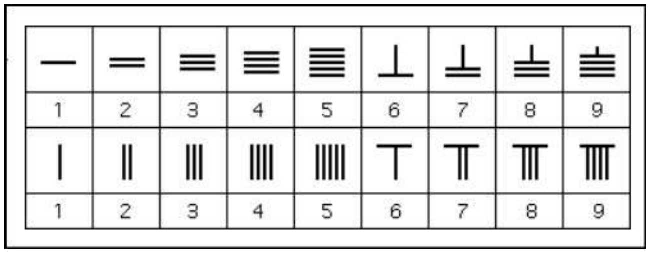
Two theories about the Chinese number system come into existence.

The first one suggests that the symbols of numerals are "phonetic". Let us consider the number 1000 symbolized by 'Man'. The word for 1000 in the Chinese number system was close to the sound of the word for man.

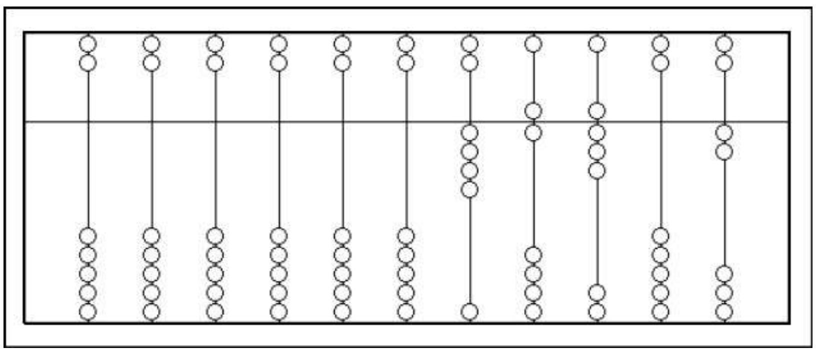
The second theory suggested that the symbols originated from the fact that
numbers were used primarily as a part of religious ceremonies

There were two other very important transformations in the Chinese Numeral
System.

• In the 4th century BC, the counting board came into use. Numerals were represented by Bamboo or Ivory and were placed in rows and columns.



• In the 14th Century, China invented "Abacus". It was a counting board that was represented by beads on a sliding wire. Abacus was used by merchants who used operations for addition and subtraction.



(IV.) GREEK NUMBER SYSTEM: *ACROPHONIC AND ALPHABETICAL GREEK SYSTEM*

Due to the Greek numeral system methodology, the basic operations and arithmetical calculations became practical. It was the most relative to humans as in the first alphabet of a number was its symbol. The word that describes this is "Acrophonic". The Greek Numeral system had different ways for two sub-systems, a cardinal number system, and an ordinal number system.

Γ	Δ	H	X	M
Pente	Deka	Hekaton	Khilioi	Murioi
Πεντε	Δεκα	Ηεκατον	Χιλιοι	Μυριοι
5	10	100	1000	10000

The alphabetical system is a sub-system of the Greek system. It is a point of consideration that Greeks were the first to adopt a numeral system based on alphabets. It comprised of 24 symbols of alphabets and 3 additional symbols.

Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Eta	Theta
Α α	Β β	Γ γ	Δ δ	Ε ε	Ζ ζ	Η η	Θ θ
1	2	3	4	5	7	8	9

Iota	Kappa	Lamda	Mu	Nu	Xi	Omicron	Pi
Ι ι	Κ κ	Λ λ	Μ μ	Ν ν	Ξ ξ	Ο ο	Π π
10	20	30	40	50	60	70	80

Rho	Sigma	Tau	Upsilon	Phi	Chi	Psi	Omega
Ρ ρ	Σ σ ς	Τ τ	Υ υ	Φ φ	Χ χ	Ψ ψ	Ω ω
100	200	300	400	500	600	700	800

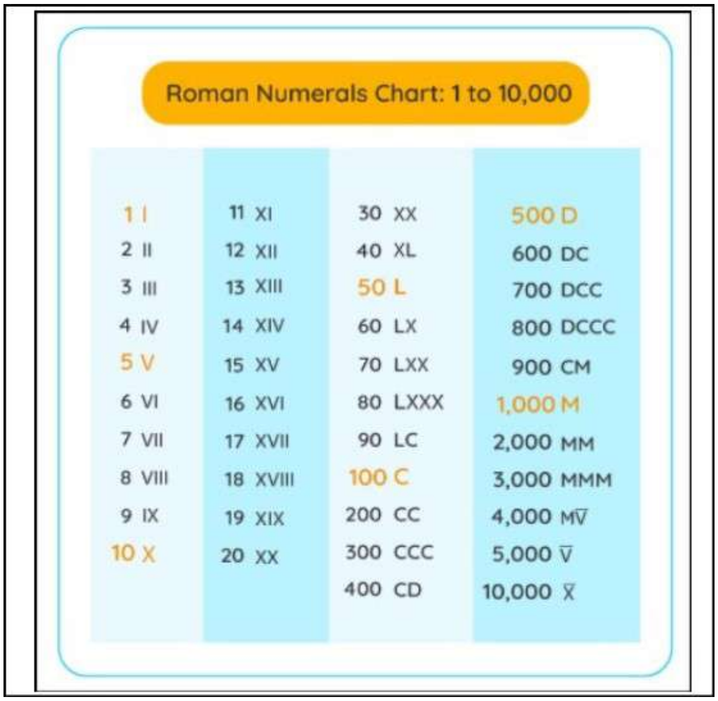
Digamma	Stigma	Koppa	Sampi
F	ϛ	ϙ	ϡ
6	6	90	900

The method to express every possible number by just using 10 symbols emerged in India after the evolution of Mathematics in its Greek understanding. The introduction and development of the Greek number system led to the facilitation of Arithmetic as an invention. The benefit of the Greek system is that zero was being recognized and was given a named place value.

(v.) Roman Number System

Latin Alphabets were required to carry out transactions and for counting purposes. These letters composed the Roman numeral system. For effective trading and communications, Roman numerals were a great invention. The aim of introducing these was that it was clumsy to count values on fingers that were greater than 10.

Historical facts suggested that Romans were known for their writing of numbers which was the reason why they had control of the empire. It was based on decimals and not place value systems like others. The Roman numerals are composed of seven symbols I, V, X, L, C, D, and M. Between 900 and 800 BC, these numerals were used for the first time.



Roman Numerals Chart: 1 to 10,000

1 I	11 XI	30 XX	500 D
2 II	12 XII	40 XL	600 DC
3 III	13 XIII	50 L	700 DCC
4 IV	14 XIV	60 LX	800 DCCC
5 V	15 XV	70 LXX	900 CM
6 VI	16 XVI	80 LXXX	1,000 M
7 VII	17 XVII	90 LC	2,000 MM
8 VIII	18 XVIII	100 C	3,000 MMM
9 IX	19 XIX	200 CC	4,000 M$\overline{V}$
10 X	20 XX	300 CCC	5,000 $\overline{V}$
		400 CD	10,000 $\overline{X}$

This decimal numeral system had an advantage over the Egyptian Numeral System. The Roman system had numeral values and symbols for 1000 and multiples of 1,00,000 though this development happened at a very late stage after civilization.
These numbers are being widely used in naming of ships, and sporting events like the Olympics. Roman numerals have various other applications too. These are used in chemistry to denote groups and periods, in manuscripts. Music theory had also employed roman numerals.

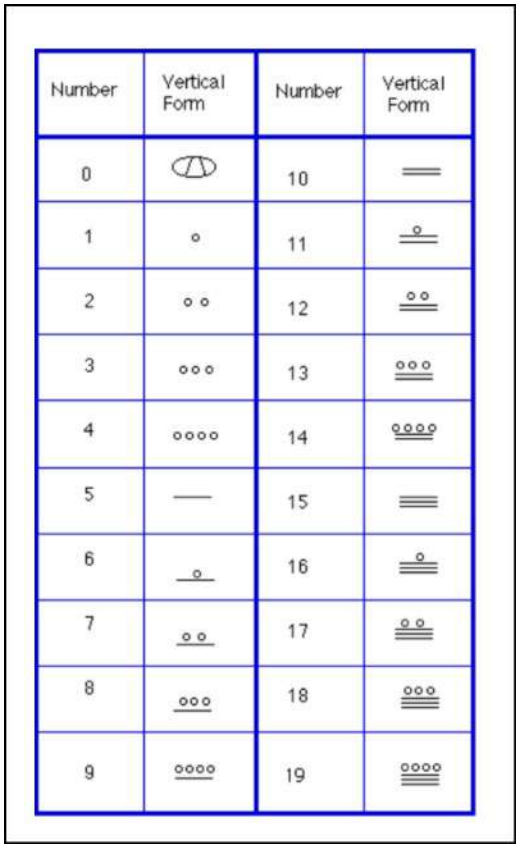
According to nature's principle, there are flaws in these numerals too. In Roman numerals, neither the fractions come into the picture nor do they have a symbol for zero.

(vi.) Mayan's Number System: *The introduction of Zero*

In between 1500BCE to 1700BCE, Mayans developed two numeral systems: one that belonged and could be understood by the common people and one for the priests. Both these systems had different symbols and base systems.

The number system that persists was the one for common people. It was called the "Vigesimal System". It had a base 20 number system. This system was represented by two symbols, a dot for 1, and a horizontal bar for 5. The uniqueness in the way of writing this number system was as follows. Numbers were represented vertically with the most significant digits on the top.

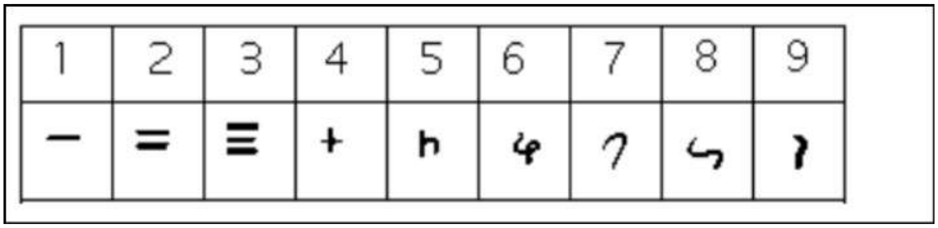
The advantages that this number system brought were that they used sticks to produce astronomical observations and measured the length of a solar year as well as the length of lunar month with a high degree of accuracy.



(vii.) Arabic or Hindu Number System

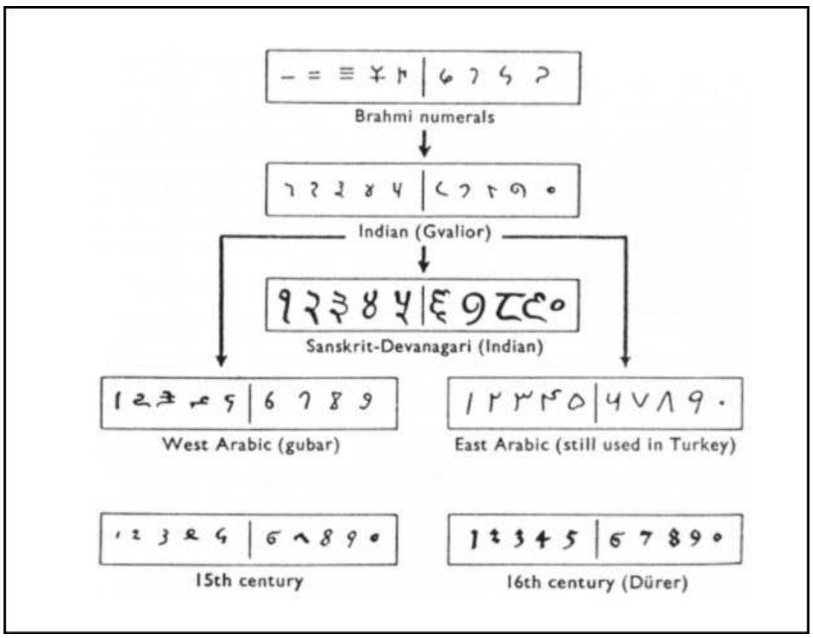
It is a decimal representation number system and consists of numbers we use at present, that is, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The invention of zero is of great significance in this numeral system. This system is based on positions. For example, in the number 239815, 3 has a much greater value, i.e., it has a place value of ten thousand than 9 which has a place value of thousand. Both, the 10 symbols and the idea of positional value are introduced in India.

Looking at historical facts, In India, complicated Brahmi numerals were used before the numbers we use at present. Until the 4th century, these numerals were used.



On the path that leads to the current number system, we also come across the Gupta Numeral system that comes into the picture during rule by the Gupta Dynasty.

There were even more stages of evolution of the number system. We had a different version of the Brahmi numerals. In the 11th century, Nagari symbols were introduced.



The Roman numeral is consistent and has proved to be persistent.

Importance of Numbers and Existing Number Systems

Beginning with counting numbers on fingers, using clay tokens and numbers taking different forms, and improving at every stage, overcoming the flaws, the number system that we use today is in its most persistent form.
The number system enables us to survive in today's world. These numbers are building blocks of this vast subject called Mathematics, and this subject has its significance in an

individual's life from the moment one takes birth. It's the first breath that one takes and initiates life, one word that an individual utters, which is of utmost satisfying factor to one's existence. A child's first and the successive steps, victories, and loss in life, bad or good times, everything involves numbers.

As a result of these numbers, we can study this vast subject of Mathematics. This subject enables us to learn the various aspects of life and help us to learn about existing structures or shapes under a sub-section known as Geometry, the art with which tells us about the position of a certain object known as trigonometry, the operations that help us to know a value called arithmetic. Like these, numbers form a base for some other branches of mathematics as well. These are: Algebra, Number Theory, Topology, Calculus, Set Theory, Statistics and Probability.

In today's time, with the advancement of technology, we are in a stage where everything moves and grows due to computers. What enables the computers to perform such functions is a beautiful creation known as programming, the existence of this technology again persists due to the presence of mathematics.

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MATHEMATICS IN MAP PROJECTION

Hariansh Suman
Year I

Abstract

This article will show some basic derivation of one of the map projections called Mercator Projection. It includes projecting grids made by latitudes and longitudes from the surface of Earth onto a plane map in a systematic manner using mathematical derivations.

Introduction

A map is a 2-dimensional image of Earth. It is commonly used by most people as it is compact and easy to store and transport. A map also shows a large portion of Earth at once. The process of making maps is called map projection which is a way to flatten Earth's surface into a plane in order to make a map. Mercator Projection is a map projection made in the late 16^{th} century but is notable and useful till date. It was the first map to represent north up and south down everywhere in the map while preserving local directions.

Mercator Projection

Mercator Projection is a cylindrical map projection, which is made by wrapping a cylinder around the globe representing the Earth. The map is an image of the globe projected onto the cylindrical surface surrounding it. Mercator Projection was created by Gerardus Mercator in 1569. Later, mathematicians like Edward Wright, Henry Bond and Thomas Harriot worked for the mathematical formulation of the map. It was one of the first maps to represent Earth as a rectangular plane with straight vertical latitudes (lines going east-west) and straight parallel longitude (lines going north-south direction) spaced evenly on the Equator.
Mercator Projection is often used in navigation, it is also used by street map services hosted on the internet due to its favorable properties of local area maps. Web Mercator Projection, a variant of Mercator Projection is used in online street mapping services like Google Maps. Map projection requires a systematic transformation of latitudes and longitudes from the globe into the locations on the plane for which mathematical derivation is required.

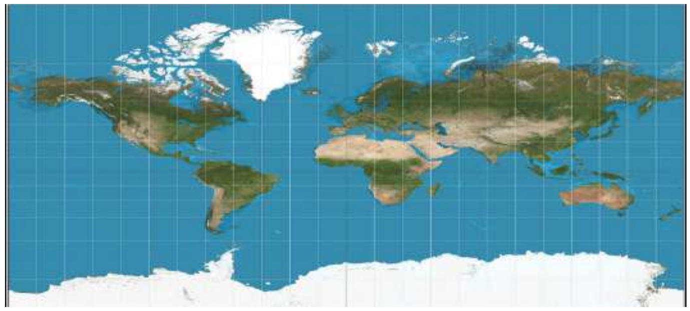

Figure 13.1

Derivation Of Formulas Required

Before starting the derivation for creating a map:

1. We need to select a model for the shape of Earth, because Earth's actual shape is irregular. Here, we are taking the shape of Earth as a sphere.
2. We need to select a surface that can be unfolded and unrolled into a plane or sheet without stretching, tearing or shrinking. Here, we are taking the projection surface as a cylinder.

Representative Fraction

As the models of Earth and projection surface have been decided, wrapping a cylinder around the Earth is not possible. So, we need to make a small image of Earth. A globe is the only way to represent the Earth with constant scale throughout the entire map in all directions. For making a globe, we use Representative Fraction (RF) expressed as a ratio $1 : X$ which shows that 1 unit on globe is x units on Earth's surface:

$$\text{RF} = \frac{1}{X} = \frac{\text{Globe Radius}}{\text{Earth Radius}} \quad \text{or} \quad \text{Globe Radius} = \frac{\text{Earth Radius}}{X} \tag{13.1}$$

Locating Points On Surface Of Globe

Now, as a small image of Earth has been made, let's take a point P on the surface of Globe. Let the angle between the latitude at P and equatorial plane (xy-plane) be angle of latitude denoted by ϕ considered in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the angle between the longitude at P and the reference meridian (established at Greenwich by the Prime Meridian Conference in 1884) be the angle of longitude denoted by λ (considered on either of the intervals $[-\pi, \pi]$ and $[0, 2\pi]$ radians). So, we represent position of point P as (λ, ϕ) and another point Q at latitude $\phi + d\phi$ and longitude $\lambda + d\lambda$ as $(\lambda + d\lambda, \phi + d\phi)$

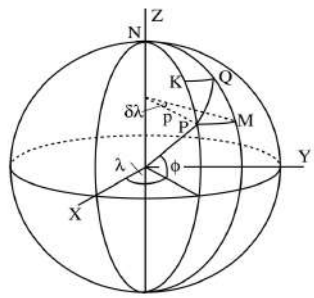

Figure 13.2

The position of point P can also be shown in xyz-plane as in figure 14.3(a), where,
R = radius of globe,
λ = angle of longitude,
p = distance from z-axis to point P,
ϕ = angle of latitude,
d = distance from xy-plane to point P

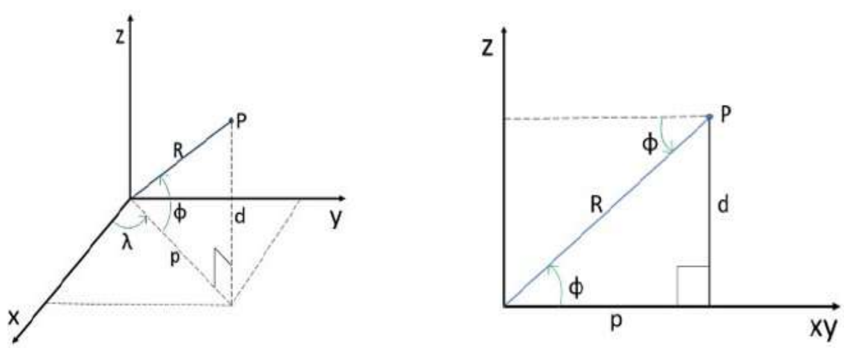

Figure 13.3: (a) and (b)

And in $xy - z$ plane, it can be shown as in figure 14.3(b). From figure 14.3(b), using Pythagoras theorem,

$$cos\,\phi = \frac{p}{R} \text{ or } p = R\,cos\phi \tag{13.2}$$

Therefore, distance of point P from z-axis can be shown as $p = R\,cos\phi$.

Cylindrical Projection

Mercator Projection is a cylindrical projection. For making a cylindrical projection of the globe representing the Earth, we place it inside a cylinder of radius same as the radius of the

globe, which is tangential to the sphere on the equator and the axis of the cylinder coincides with the polar diameter (diameter between the poles).
For making the map, we take a plane at the axis that intersect the sphere at two longitudes at $\lambda = 0$ and $\lambda = 180°$ *i.e* π

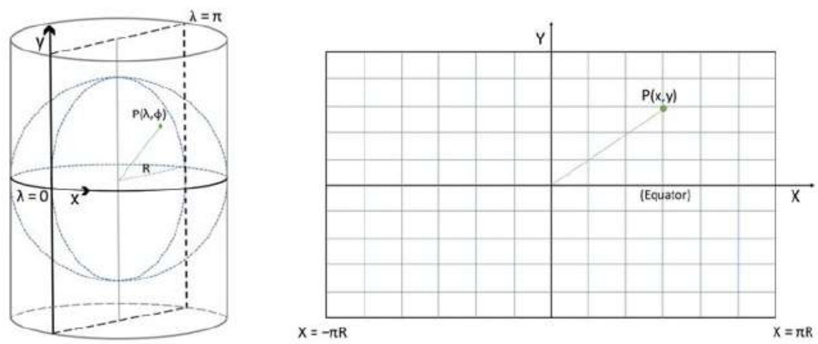

Figure 13.4: (a) and (b)

The cylinder is than cut along the longitude $\lambda = \pi$ and unrolled. The axis on the map is chosen as x-axis along the equator and y-axis coincident with the plane at $\lambda = 0$. And the point P on the map can be shown with coordinates (x, y).
As the radius of sphere is taken R, the length of equator will be equal to the circumference of the circle of radius R. The length of equator will be equal to $2\pi R$ and also equator is same as x-axis, so, length of x-axis will be equal to $2\pi R$ or x-axis in positive and negative direction goes up to $\frac{2\pi R}{2} = \pi R$. As πR is the end point of the axis, which is the product of angle of longitude and the radius, the equation for x-axis can be given by $x = R\,\lambda$.
For the equation of y-axis on the projection, we take product of radius and any function of ϕ,
the equation for y-axis can be given by $y = R\,f(\phi)$
Therefore, the equation for finding x and y coordinates can be given as

$$x = R\,\lambda$$

$$y = R\,f(\phi)$$

where λ and ϕ are in radians.

Conformality

One of the important properties of Mercator Projection is that it is a conformal map *i.e.*, a map in which angle between two curves that cross each other on Earth is preserved on the map.
Now, let's take an area element of the sphere as a grid made by latitude and longitude as $PMQK$ which is a trapezium. As we can see in figure 14.5(a), angle PKE_1 and angle

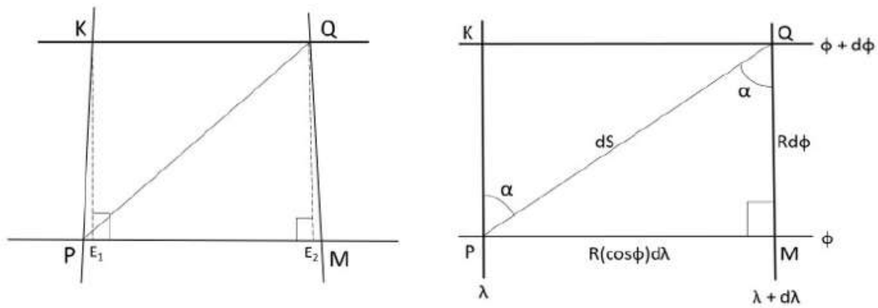

Figure 13.5: (a) and (b)

MQE_2 are very small, so, we can take the area element as a rectangle as shown in figure 14.5(b).
Let PK is on longitude λ and MQ is on longitude $\lambda + d\lambda$, similarly, let PM is on latitude ϕ and KQ is on latitude $\phi + d\phi$.
From triangle PMQ,

$$\tan \alpha = \frac{PM}{QM} \tag{13.3}$$

As PM and KP are actually arcs whose length can be found using arc length formula,

$$\text{Arc length} = \text{Radius} \times \text{Angle}$$

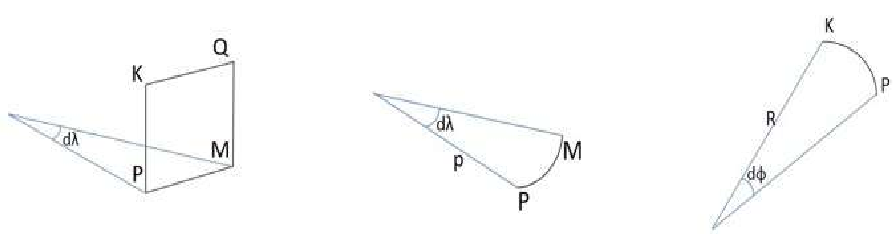

Figure 13.6: (a), (b) and (c)

Now, in figure 14.6(a), the difference between angle of longitude of PK and QM is $d\lambda$ and the distance of point P from the centre axis (z-axis) is p. As we are taking a small area element, so we can say that the difference of point M from z-axis is also p. Similarly in figure 14.6(c), the angle is $d\phi$ and the distance of points K and P from the centre is R.
Now, using arc length formula in figure 14.6(b) and 14.6(c), $PM = pd\lambda$ and $PK = Rd\phi$
From equation (14.2),

$$p = R\cos\phi$$

So,

$$PM = R\cos\phi d\lambda$$

and also,

$$PK = QM$$

So,

$$QM = Rd\phi$$

Therefore, equation (14.3) can be written as,

$$\tanh \alpha = \frac{R \cos \phi \, d\lambda}{R \, d\phi} \tag{13.4}$$

Now, lets take an area element of the cylinder on which map is to be projected as P'M'Q'K' as a rectangle where P'K' on x-axis is at point x and M'Q' is at point $x + dx$. Let P'M' on y-axis is at point y and K'Q' at point $y + dy$

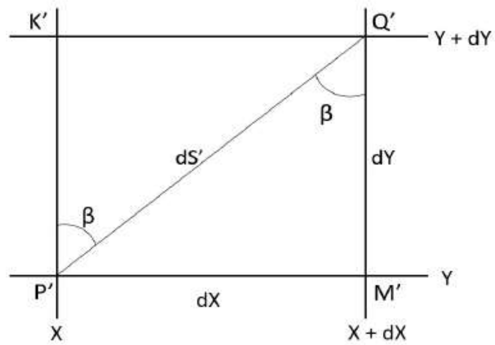

Figure 13.7

Now from triangle P'M'Q', $\tan \beta = \frac{dx}{dy}$
As $x = R\lambda$
differentiating x w.r.t. λ,
$\frac{dx}{d\lambda} = 1$ or $dx = d\lambda$
Similarly, $y = Rf(\phi)$
differentiating y w.r.t. ϕ,
$\frac{dy}{d\phi} = f'(\phi)$ or $dy = f'(\phi)d\phi$
So, $\tan \beta$ can be written as,

$$\tan \beta = \frac{d\lambda}{f'(\phi)d\phi} \tag{13.5}$$

From equation (4) and (5), relation between $\tan \alpha$ and $\tan \beta$ can be given as,
$\tan \alpha = \cos \phi f'(\phi) \tan \beta$ or $\tan \beta = \frac{sec\phi}{f'(\phi)} tan\alpha$
As mentioned earlier, Mercator Map is a conformal map in which angles are preserved. For our map projection to be conformal, α should be equal to β or $\tan \alpha = \tan \beta$.
As $\tan \beta = \frac{sec\phi}{f'(\phi)} tan\alpha$
So, the condition for the map to be conformal is $\frac{\sec \phi}{f'(\phi)} = 1$ or $f'(\phi) = \sec \phi$

Finding $f(\phi)$:

As the equation for finding coordinates of y-axis depends on $f(\phi)$.So, we need to find the function $f(\phi)$ by integrating $\sec\phi$ with limit 0 to π.

$$f(\phi) = \int_0^\pi \sec\phi d\phi \tag{13.6}$$

As $\cos\phi = \sin(\phi + \frac{\pi}{2})$,
Also $\sin 2x = 2\sin x \cos x$ Or $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$
So, $\sin(\phi + \frac{\pi}{2})$ can be written as $2\sin(\frac{\phi}{2} + \frac{\pi}{4})\cos(\frac{\phi}{2} + \frac{\pi}{4})$
Or $\cos\phi = 2\sin(\frac{\phi}{2} + \frac{\pi}{4})\cos(\frac{\phi}{2} + \frac{\pi}{4})$
Dividing and multiplying R.H.S. by $\cos(\frac{\phi}{2} + \frac{\pi}{4})$

$$\cos\phi = 2\sin\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\cos\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\frac{\cos(\frac{\phi}{2} + \frac{\pi}{4})}{\cos(\frac{\phi}{2} + \frac{\pi}{4})} = 2\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\cos^2\left(\frac{\phi}{2} + \frac{\pi}{4}\right)$$

As,

$$\sec\phi = \frac{1}{\cos\phi} = \frac{1}{2\tan(\frac{\phi}{2} + \frac{\pi}{4})\cos^2(\frac{\phi}{2} + \frac{\pi}{4})} = \frac{\sec^2(\frac{\phi}{2} + \frac{\pi}{4})}{2\tan(\frac{\phi}{2} + \frac{\pi}{4})}$$

So, equation (6) can be written as

$$f(\phi) = \int_0^\pi \frac{\sec^2(\frac{\phi}{2} + \frac{\pi}{4})}{2\tan(\frac{\phi}{2} + \frac{\pi}{4})} d\phi = \int_0^\pi \frac{\frac{\sec^2(\frac{\phi}{2} + \frac{\pi}{4})}{2}}{\tan(\frac{\phi}{2} + \frac{\pi}{4})} d\phi$$

Let $u = \tan(\frac{\phi}{2} + \frac{\pi}{4})$
Differentiating u w.r.t.ϕ

$$\frac{du}{d\phi} = \frac{1}{2}\sec^2\left(\frac{\phi}{2} + \frac{\pi}{4}\right) \text{ or } du = \frac{1}{2}\sec^2\left(\frac{\phi}{2} + \frac{\pi}{4}\right)d\phi$$

so, $f(\phi)$ can be written as

$$f(\phi) = \int_0^\pi \frac{du}{u} = \ln|u| \text{ or } f(\phi) = \ln|\tan(\frac{\phi}{2} + \frac{\pi}{4})|$$

Therefore, the mathematical formula for Mercator Projection can be given as Mercator Projection in Normal Equatorial aspect can be given as,
$X = R\lambda$
$Y = R\,f(\phi)$
where, $f(\phi) = \ln|\tan(\frac{\phi}{2} + \frac{\pi}{4})|$

Results

Now, as the formula for Mercator projection has been derived, we can make graticule of Mercator Projection of the Earth.

Firstly, for making a small globe, Let us take the Representative Fraction as 1:182228571

As the equatorial Radius of Earth is 6378 km = 637800000 cm

So, the radius of the globe is $= \frac{\text{Radius of Earth}}{182228571} = \frac{637800000}{182228571} = 3.5\,\text{cm}$

Now, length of the equator = length of x axis $= 2\pi r = 2 \times \frac{22}{7} \times 3.5 = 22\,\text{cm}$

And separately positive and negative = 11 cm

As $x = R\lambda$, where $\lambda =$ longitude

For $\lambda = 30° = \frac{\pi}{6},\ x = 3.5 \times \frac{\pi}{6} = 1.83\,\text{cm},$
For $\lambda = 60° = \frac{\pi}{3},\ x = 3.5 \times \frac{\pi}{3} = 3.64\,\text{cm},$
For $\lambda = 90° = \frac{\pi}{2},\ x = 3.5 \times \frac{\pi}{2} = 5.5\,\text{cm},$
For $\lambda = 120° = \frac{2\pi}{3},\ x = 3.5 \times \frac{2\pi}{3} = 7.3\,\text{cm},$
For $\lambda = 150° = \frac{5\pi}{6},\ x = 3.5 \times \frac{5\pi}{6} = 9.16\,\text{cm},$
For $\lambda = 180° = \pi,\ x = 3.5 \times \pi = 11\,\text{cm},$

Now, as the equation for finding $f(\phi)$ for interval of $20°$

$$\text{For } \phi = 20° = \frac{\pi}{9},\ f(\frac{\pi}{9}) = \ln\left|\tan\left(\frac{\frac{\pi}{9}}{2} + \frac{\pi}{4}\right)\right| = \ln\left|\tan\left(\frac{11\pi}{36}\right)\right| = \ln|1.4281480| = 0.356$$

$$\text{For } \phi = 40° = \frac{2\pi}{9},\ f(\frac{2\pi}{9}) = \ln\left|\tan\left(\frac{\frac{2\pi}{9}}{2} + \frac{\pi}{4}\right)\right| = \ln\left|\tan\left(\frac{13\pi}{36}\right)\right| = \ln|2.14450692| = 0.763$$

$$\text{For } \phi = 60° = \frac{\pi}{3},\ f(\frac{\pi}{3}) = \ln\left|\tan\left(\frac{\frac{\pi}{3}}{2} + \frac{\pi}{4}\right)\right| = \ln\left|\tan\left(\frac{5\pi}{12}\right)\right| = \ln|3.7320508| = 1.317$$

$$\text{For } \phi = 80° = \frac{4\pi}{9},\ f(\frac{4\pi}{9}) = \ln\left|\tan\left(\frac{\frac{4\pi}{9}}{2} + \frac{\pi}{4}\right)\right| = \ln\left|\tan\left(\frac{17\pi}{36}\right)\right| = \ln|11.4300523| = 2.436$$

Now, for finding y coordinates,

$$\text{For } \phi = 20^\circ,\ y = R\,f(20) = 3.5 \times 0.356 = 1.25\,\text{cm}$$

$$\text{For } \phi = 40^\circ,\ y = R\,f(40) = 3.5 \times 0.763 = 2.67\,\text{cm}$$

$$\text{For } \phi = 60^\circ,\ y = R\,f(60) = 3.5 \times 1.317 = 4.61\,\text{cm}$$

$$\text{For } \phi = 80^\circ,\ y = R\,f(80) = 3.5 \times 2.436 = 8.53\,\text{cm}$$

As, we have found the scale of x and y axis, the grids of latitudes and longitudes on the map can be represented as

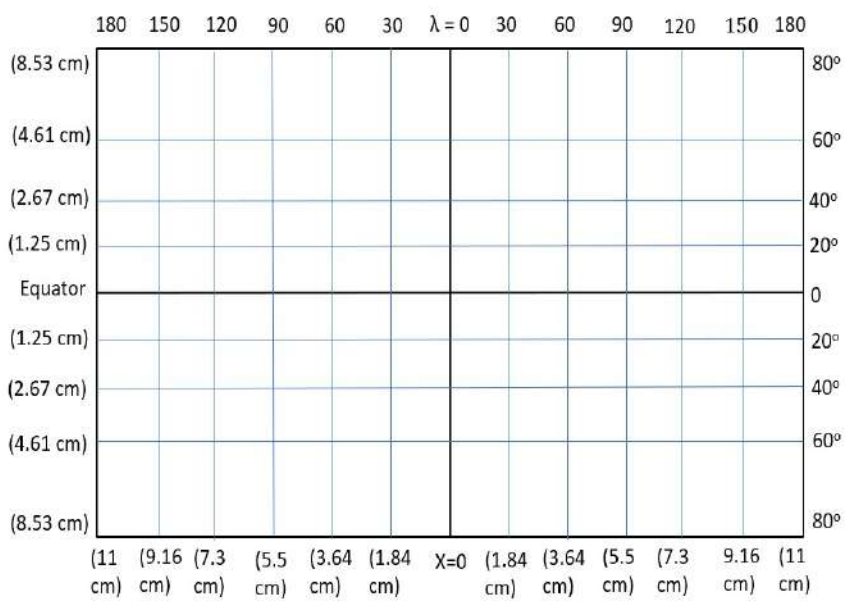

Figure 13.8

As the map has been made as a 2-dimensional cartesian plane, any point P on the globe surface with angle of latitude and longitude as λ and ϕ, can also be located on the map as,

$$P(\lambda, \phi) = P(x, y),\ X = R\,\lambda,\ Y = R\,f(\phi)$$

Or

$$P(\lambda, \phi) = P(R\,\lambda, R\,f(\phi))$$

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ON CARDINALITIES OF INFINITE SETS

Abstract
In this paper we discuss the cardinalities of infinite sets and give an intuitive explanation for how cardinalities of natural numbers and integers are the same when it would seem that integers are twice the size of natural numbers. We accomplish this by defining the term 'cardinality' in mathematics, providing an essence of counting and discuss how usual arithmetic does not apply to infinities. We also look into Cantor's Continuum Hypothesis.

INTRODUCTION

Infinity is not exactly a number. It's an idea used to express how endless things can be. When we hear infinity, the first thought that strikes our mind is that 'infinity is the limit' which is exactly opposite of it. Infinity is here to give us a sense of limitlessness and not to define a very big limit. But how big is it? And are all infinities the same? We'll explore this idea by discussing infinite sets and their cardinalities and also how exactly some infinities are bigger than others.

WHAT IS CARDINALITY

Cardinality in mathematics describes the 'size' of a set. If the set is of finite number of elements and is small enough, you can simply count the number of elements to determine its cardinality.
For example : $A = \{x : x \in \mathbb{N}, x \leq 100, x \text{ is a perfect square }\}$.
As the number of elements in set A is countable, we can simply state its cardinality as $n(A) = 10$.

But what do we do when the sets have an infinite number of elements? In such cases, we use a different approach. As the number of elements are infinite, it is clear we cannot state the exact size of the set so we compute the size of these sets using bijections. So if we are able to define a bijective function from set 1 to set 2 both of infinite cardinalities, we are able to prove that both sets have the same cardinalities.

Intuitive explanation of cardinalities: As mentioned earlier in the paper, cardinality is defined as ‘size’ of a set. When the set is finite, we can simply count and establish its cardinality. While establishing the size of a finite set, we are using the concept of counting, which in its essence is establishing a one-to-one correspondence[1].

How? For example, before the concept of counting came into being, when a farmer had to keep checks of his number of cattle, he’d keep one pebble for each cattle and later when they returned he’d make sure that there still was one pebble for each cattle. By doing this, the farmer was basically establishing a bijection between the number of pebbles and the number of cattle. This is the technique that was employed to count before natural numbers were defined. And this is basically the essence of counting as well. Just that now instead of objects (like the pebbles here), we have the set of Natural Numbers to establish this bijection. Now you must be thinking why do we need this information to know about the cardinalities? Well, while defining the cardinality of finite sets, we have natural numbers as a tool to give an idea of the exact number of quantities for the size of a set.

Things are different when it comes to infinite sets because of two things:

1. There is no natural number in the number system that can be used to define cardinality of an infinite set. Hence, instead of finding the exact size, we actually compare the cardinalities of these sets and determine if they are equal or not.

2. As natural numbers cannot be used, it can be concluded that ‘counting’ is not the technique to determine the size of these sets. Hence, the only and most basic thing we’re left with is establishing a bijection to ‘compare’ the cardinalities. We do this by defining a function and proving it is bijective as mentioned earlier.

COMPARING CARDINALITIES OF DIFFERENT SETS

In this section we’ll try to explain how (‘intuitively’) the set of natural numbers and integers have the same cardinalities and how real numbers have a greater cardinality than natural numbers.

Natural Numbers and Integers: We know by bijection that natural numbers and integers have the same cardinalities. We may have proved it mathematically but when we think about it, it is very natural to wonder why aren’t integers twice as natural numbers?

To answer this question, we have to realize that it is already given that both the sets have infinite cardinalities and that the fundamental operations in the real number system such as addition and multiplication do not work for infinities. So, even if we might think that natural numbers have size infinity and hence, integers should have size, let us say, $\infty + \infty = 2\infty$. THis is false because arithmetic operations do not work for infinities and hence, this logic cannot be used. Thus, the only way we can get a sense of size of an infinite set is via comparing it with size of other Infinite sets.

Natural Numbers and Real Numbers: We now know about the cardinalities of Natural numbers and integers and how they are equal and countable. What about real numbers? There does not exist a bijection from $\mathbb{N}$ to $\mathbb{R}$. That is, Real numbers have a different cardinality. We won't be looking at the exact proof but will see how and why this is.

Let's look at a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ which assigns a real number to each natural number[2].

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

Now we take the diagonal elements into consideration and add 1 to each element.i.e.,

Add 1 to the first digit of the first entry
Add 1 to the second digit of the second entry and so on

0.5161766..

This will construct a number $b \in \mathbb{R}$ that does not have a corresponding natural number to it. What we've concluded is that there does not exist a surjection from $\mathbb{N}$ to $\mathbb{R}$ i.e., $\mathbb{R}$ has a greater cardinality than N.[2]

So $\mathbb{R}$ has a greater cardinality. But is there a set with greater cardinality than $\mathbb{R}$? The answer is yes. The power set of $\mathbb{R}$. Power set of any set has a greater cardinality than the set itself and using this we can obtain an infinite number of sets, each having greater cardinality than the previous one. Infinitely many infinities!

CANTOR'S CONTINUUM HYPOTHESIS

Thus, the cardinality of $\mathbb{N}$ is less than the cardinality of $\mathbb{R}$. Is there a set with an intermediate cardinality? That is, a set with cardinality greater than that of $\mathbb{N}$, but less than that of $\mathbb{R}$. Georg Cantor put forth a Hypothesis that said the answer is no. He argued that there does not exist an infinite set of real numbers that could be put into one-to-one correspondence neither with the natural numbers nor with the real numbers[3]. But he couldn't prove it and it is unsolvable i.e., one cannot solve it.

CONCLUSION

We saw how counting is basically establishing a bijection and therefore even in the problems where natural numbers fail to help us in determining the size of a set, we can employ the method of bijections to compare the cardinalities of different sets. We also discussed an intuitive explanation for why set of natural numbers and integers have same cardinalities and caught a glimpse of Cantor's Continuum Hypothesis.

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ANTI PASCAL TRIANGLE

From the Notebooks of Evan Chen and Russelin

Abstract

The IMO is a celebration of mathematics' beauty and people's innovative problem-solving abilities. Problems aren't everyone's favourite thing, are they? And the IMO problems are especially challenging. After all, they must confront the world's greatest young minds. But that's all there is to it. Problems are appealing because they present a challenge. And the pleasure derived from solving issues is undeniable. In this paper we aim to decipher one such question which appeared in one of the world's toughest Mathematics Olympiads and try to reach out the most common interpretation to this.

IMO 2018 Problem 3

An anti-Pascal triangle is an equilateral triangular array of integers where each number is the absolute value of the difference of the two numbers immediately below it, except for the numbers in the bottom row. The following array, for example, is an anti-Pascal triangle with four rows containing each integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 4 & & 7 & & \\ & 5 & & 9 & & 2 & \\ 6 & & 1 & & 10 & & 8 \end{array}$$

The question is: **Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from** 1 to $1 + 2 + \cdots + 2018$?

Pascal's Triangle

The coefficients in the expansion of every binomial equation are given by Pascal's Triangle, a triangular arrangement of integers. It is named after the 17^{th} century French mathematician Blaise Pascal, who wrote the Treatise on the Arithmetical Triangle. It is now known as the Pascal's Triangle, but it's actually much older. A triangular representation for the coefficients was devised by Chinese mathematician Jia Xian in the 11^{th} century. Yang Hui, a Chinese mathematician, popularised his triangle in the 13^{th} century.

The numbers are placed in a triangle-like pattern. We start by placing 1 at the top and working our way down in a triangle manner. The sum of the two numbers immediately above a spot is the number we acquire in each stage.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \end{array}$$

Formula

The entry in the p^{th} row and the q^{th} column of Pascal's triangle is denoted by the Binomial Coefficient: $\begin{pmatrix} p \\ q \end{pmatrix}$.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p-1 \\ q-1 \end{pmatrix} + \begin{pmatrix} p-1 \\ q \end{pmatrix}$$

for any non-negative integer p and any integer $0 \le x \le p$.

Binomial Expansion

The coefficients that appear in binomial expansion are determined by Pascal's triangle.

$$\begin{aligned} (x+y)^2 &= x^2 + 2xy + y^2 \\ &= \mathbf{1}x^2y^0 + \mathbf{2}x^1y^1 + \mathbf{1}x^0y^2 \end{aligned}$$

The coefficients are the numbers in the second row of Pascal's triangle:
$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2$, $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
In general,

$$\begin{aligned} (x+y)^n &= \sum_{k=0}^{n} a_k x^{n-k} y^k \\ &= a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n \end{aligned}$$

where the coefficient a_k in this expansion is precisely the numbers on the row n of Pascal's Triangle i.e. $a_k = \binom{n}{k}$.

Anti Pascal Triangle

A finite set of numbers arranged in a triangle-shaped array, with one number in the first row, two numbers in the second row, three numbers in the third row, and so on; and each number is equal to the absolute value of the difference of the two numbers below it, except for the numbers in the bottom row, is called an Anti-Pascal Triangle.

Solution 1

Let us consider an anti-Pascal triangle with 4 rows.

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 4 & & 7 & & \\ & 5 & & 9 & & 2 & \\ 6 & & 1 & & \mathbf{10} & & 8 \end{array}$$

Here, the largest number 10 is in the bottom row.
The smallest numbers 1, 2, 3 and 4 are placed in such a way that they contribute to sum 10 in bottom.
Concretely speaking, if we create a path from top to 10, passing through the maximum number of each row, then the numbers 1, 2, 3 and 4 lie adjacent to this 'path to 10'.
Knowing the position of the small numbers eliminates the possibility of other sections of the triangle containing small numbers. This is the key to demonstrate that a 2018 row anti Pascal pyramid does not exist.

Let's now move on to 2018 row triangle. Let $M = 1 + 2 + 3 + \cdots + 2018$ be the greatest number in our presumably appropriate triangle. M will be placed in the bottom row, a diagram is illustrated here highlighting the "route to M", which has all the numbers from 1 to 2018 adjacent to it.

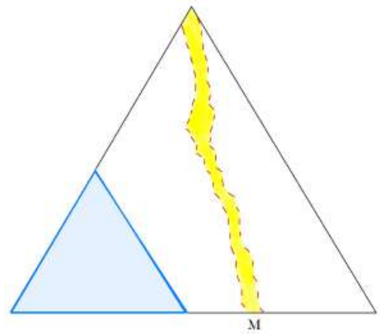

Figure 15.1

The "route to M" is marked in yellow in the diagram above. Pink dots next to the yellow path symbolise "small numbers" in this case numbers up to 2018.
We may find a triangle [shown in blue in the figure above] at the bottom right or left corner whose digits must all be 2019 or more, since the number M must be to the left or right of the centre on the bottom row.
Because there must be at least 1008 $\left(\frac{n}{2}-1\right)$ rows in this triangle. The minimum feasible total is

$$S = 2019 + 2020 + \cdots + (2019 + 1007)$$

But the maximum number for our anti-Pascal triangle is

$$\begin{aligned} M &= 1 + 2 + \cdots + 2018 \\ &= \frac{2018(2018+1)}{2} \end{aligned}$$

which shows that $S > M$ However, a number greater than M cannot be contained within the triangle. As a result, the hypothetical anti-Pascal pyramid with 2018 rows is not possible.

Solution 2

Let $n = 2018$ and $M = 1 + 2 + 3 + \cdots + n$
For any number 'a' (not in the bottom row), Draw an arrow from a to larger of the 2 numbers below it (*i.e* if $a = x - y$ draw $a \to x$. This creates lightning strikes.
Consider the directed path that begins at the top vertex A and ends at the bottom vertex B. It increases by at least $1 + 2 + \cdots + n$ starting from the first number because the increments at each step in the path are distinct; Consequently, equality must hold, and the path from top ends at $M = 1 + 2 + \cdots + n$ where all the numbers $1, 2, \cdots, n$ are adjacent to the path created.

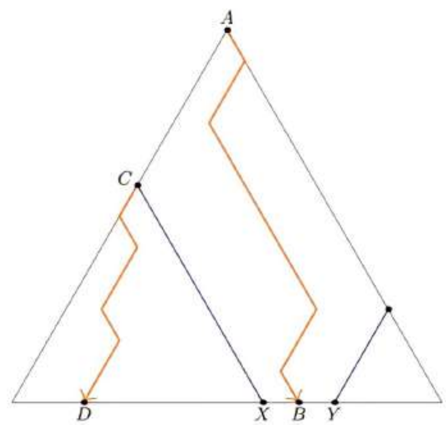

Figure 15.2

Consider the endpoint B's left and right neighbors *i.e.,* X and Y Assuming B to be at the right of the bottom side's midpoint, We finish the equilateral triangle to an open C as illustrated. Considering a lightening bolt from C that land at D's bottom.

By construction, it transverses at least $\left(\frac{n}{2} - 1\right)$ steps. However, because $1, 2, \cdots, n$ are close to $A \to B$ the lightening path, the increments must be at least $n+1, n+2, \cdots$ From this, we conclude that D must be at least

$$(n+1) + (n+2) + \cdots + \left(n + \left(\frac{n}{2} - 1\right)\right)$$

which is greater than $1 + 2 + \cdots + n$ for $n \geq 2018$.

This is a contradiction as the number at D is greater than the number at B which should be maximum. Thus, there exists no such anti-Pascal triangle.

Conclusion

The beauty of this problem is that it is both incredibly difficult and deceptively simple. It makes all the difference to be able to perceive this problem from a slightly different perspective.

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